Error Analysis

Tyson Clarke, Chris Johnson, Cigdem Ozkan
Department of Physics, University of Utah, Salt Lake City, UT 84112

Abstract

Students in undergraduate physics laboratories work on experiments in which they are required to perform error analysis on their measured data. Error analysis is a requirement for these labs. However, error analysis has not been emphasized in previous courses. Brief explanations of error analysis are provided in the laboratory handouts, but these handouts do not explain error analysis sufficiently.

In our paper we briefly explain why error analysis is important, and how to perform error analysis. Concepts, especially the concept of the propagation of errors, are clarified with several examples.

Introduction

If you are a student beginning lab work in the undergraduate physics labs, this paper is for you. The purpose of this paper is to show you how to perform certain calculations that will be required in the lab reports you write.

You will write a lab report for each experiment you perform. In your reports you will be expected to perform error analysis. Error analysis does not mean you analyze your report for spelling or punctuation mistakes. Instead, error analysis means you analyze uncertainties in your measurements and conclusions. Uncertainties and their propagation through measurements will be explained in this paper through many examples. The highest-level mathematics required to understand this paper is basic calculus, namely taking derivatives.

Types of errors encountered in the lab

When you work in the freshman physics labs, you will encounter random error and systematic error in all of your measurements.

Random error

Random error is caused by uncontrollable conditions in the experiment. For example, you can repeatedly measure the time it takes for a cart to roll down an inclined plane. Each time you repeat the measurement, you notice that you have a slightly different time measurement. The differences in the measurements of time could be caused by several factors. For example, your physical reaction time could contaminate your time measurements or small irregularities in the inclined plane may affect the elapsed time. Such errors are random and impossible to eliminate completely. You can only hope to minimize the effects of random error in your experiments by taking careful measurements or improving your experimental techniques.

Systematic error

Systematic error is caused by faulty experimental methods, by uncalibrated equipment, or by some other variable that consistently contaminates the measurements. In the example of the cart rolling down the inclined plane, a slow stopwatch would consistently give you wrong times. Systematic errors will always modify your measurements by the same amount every time you perform that measurement. In the inclined plane example, more systematic error could be introduced if:

- The entire building was not level, thus altering the true slope of the plane.
- The inclined plane was not really straight, but bowed or bent.

Eliminating systematic error, such as those shown above is a very difficult process.

Since you cannot get rid of all error in the lab, you need to report errors in your measurements. These errors can be reported as uncertainties in each measurement you make.

This paper is written with the assumption that error and uncertainty is the same thing. These two words are synonyms, and this paper will use error and uncertainty interchangeably.

Measuring Uncertainties

Uncertainties exist in any measurements you make, no matter how carefully you measure. For instance, you could measure the height of a door with a tape measure. The markings on the tape measure might not line up exactly with the top of the door. The height of the door is uncertain because it does not fall exactly on one of the markings on the tape. If you measure the height of the door with a tape with finer markings, an uncertainty would still exist. No matter how carefully you perform the measurements with any instrument you use, you will never completely eliminate the uncertainty. Since uncertainties always remain, you will need to report them in your measurements. The uncertainties will be used to check the reliability of your data. Reporting uncertainties also allows other people to interpret the quality of your measurements and place confidence in the data you report.

Uncertainties can be estimated from electronic and physical scales, such as rulers, calipers, digital and analog scales.

Physical scales

Physical scales such as, rulers or calipers, have graduated markings that represent different amounts of the quantity being measured. Imagine you want to measure the length of a pencil by using a ruler marked in millimeters. You observe that the end of the pencil lies on the 50 millimeter (mm) mark. You can state with confidence that your measurement is closer to 50 mm than to 49 or 51 mm. Therefore your best estimate of the length is 50mm, with a probable range of 49 to 51 mm. The probable range is reported in shorthand form as an uncertainty around the best estimate like this:

$$Length = 50 \pm 1\, mm.$$

If this error or uncertainty is still too large for your purposes you can guess, at a finer measurement. After looking at the pencil more closely, you see it is a tiny bit longer than 50 mm. Suppose you can estimate down to a fraction of a millimeter. In our example, if you wanted to be cautious you could estimate the length at 50.5 mm, with an error of 0.5 mm. You would then report the length to be

$$Length = 50.5 \pm 0.5\, mm.$$

The estimating of uncertainties, finer than the smallest markings on a ruler, is called interpolation. You can improve your interpolation techniques with practice.

Digital scales

Imagine you want to measure the electric resistance of a resistor by using an electronic multimeter. The screen that displays the resistance values can only hold a few digits. Therefore the digital multimeter will round the last digit and display it on the screen. Rounding the last digit will cause an error in your measurement. Suppose the digital multimeter reads out a display of 7.84. You can be reasonably sure that the actual resistance lies between 7.83 and 7.85. You need only consider the last digit of the display when estimating your uncertainty. A shorthand way of writing this estimated uncertainty is 7.84 ± 0.01. The uncertainty in any measurement x is usually denoted by the Greek symbol *delta* (δ). Therefore, the uncertainty of the measurement x is written δx. In the case of the measured resistance, the uncertainty of the resistance δR is 0.01.

Note: *This method of estimating uncertainties from a digital scale is only valid if you have not been given the actual uncertainty of the scale in its operation manual.*

Now that you have an idea how to estimate the uncertainties in a measurement from a digital or physical scale, you need to know how to report these uncertainties with an appropriate number of significant figures.

Significant figures

The issue of significant figure arises when you have to shorten a decimal number to an appropriate size. You must then ask, “What is the appropriate size?” For instance, the number π is a decimal number with an infinite number of digits after the decimal. You cannot write all of these digits while making calculations. Therefore, if you want three significant figures, you round π to 3.14.

When reporting uncertainties, the decimal position of the uncertainty, *must* match the decimal position of the last significant digit of the measured value. For example, it would be ridiculous if you measured 5.1 centimeters and you stated your uncertainty as 0.08157987. There is no conceivable way you could have estimated the uncertainty to such a fine degree. You should round the error to one decimal place, to match the decimal place of the last significant figure in the stated measurement, like this:

$$5.1 \pm 0.1\,cm\ .$$

The example given above shows the inappropriateness of stating too much precision in the uncertainty. Now we will discuss matching significant figures in measurements and uncertainties. For example,

- Suppose your answer is 258.5 with an uncertainty of 0.531. The answer should be written as:

$$258.5 \pm 0.5\,.$$

- If the same answer has an uncertainty of 5, then answer should be written as:

$$259 \pm 5\,.$$

- If the same answer has an uncertainty of 50, then answer should be written as:

$$260 \pm 50\,.$$

Warning!!! *Many professors have strong opinions about significant figures. You should talk to your professor about how many significant figures need to be reported in your measurements.*

How uncertainties propagate through many measurements and calculations

In many experiments in the lab, you will be asked to calculate a quantity that is composed of several variables that you can measure directly. For example, you might be calculating the electrical resistance of some resistor by measuring the voltage across the resistor and the current through the resistor. The uncertainty of each measured variable will contribute to the final uncertainty in the quantity being calculated. This is called error propagation, and is something that you will be required to calculate.

There are two common cases of error propagation. The first case is the propagation of uncertainties in a quantity composed of a sum or difference of measured variables. The second case is the propagation of uncertainties in a quantity composed of a product or quotient of several variables. After discussing these two common cases, we will present a general formula that can be used to calculate the propagation of uncertainties composed of *any* combination of several measured variables.

Propagation of uncertainties in sums and differences

In the lab you will be required to calculate the final uncertainty of a quantity q that is the sum of two direct measurements x and y. If you are asked to calculate the final uncertainty of q, you might consider adding the uncertainties of x and y. This summation is not completely correct. Adding the uncertainties this way might overestimate the final uncertainty of the quantity. For example, suppose you are stacking 1000 bricks and each brick weighs 10 lbs, plus or minus 0.1 lbs. Your stack will weigh around 10,000 lbs, but you need to write down the uncertainty. You could say that the uncertainty is 100 x 0.1 lbs = 100 lbs, so that the stack will weigh 10,000 plus or minus 100 lbs. But this really represents a sort of worst-case scenario, the case that ALL the bricks are 0.1 lbs too heavy or 0.1 lbs too light. It is much more likely that some bricks are too heavy and some bricks are too light. There will then be some cancellation of the 'heavies' and 'lights' so that the stack will not be so far off as the worst-case scenario. It turns out that a useful way of estimating the uncertainty in such a case is by "adding in quadrature". Adding in quadrature is taking the square root of the sum of the squares of the uncertainties.

Here is an example of adding in quadrature. Consider the case of uncertainty propagation in a quantity q composed of the sum of two measurements x and y with uncertainties δx and δy.

$$x = 7.65 \pm 0.05\, m \text{ and}$$
$$y = 3.42 \pm 0.03\, m\ .$$

You are then asked to calculate $q = x + y$ and report the uncertainty δq. You calculate

$$q = x + y = (7.65 + 3.42) m = 11.07\, m\ .$$

You calculate δq by adding the uncertainties of x and y in quadrature like this

$$\text{Final uncertainty of } q = \delta q = \sqrt{(\delta x)^2 + (\delta y)^2}\ .$$

When you add uncertainties in quadrature, you will always end up with a result less than the usual sum. In this way you will avoid overestimating the error. In other words,

$$\sqrt{(\delta x)^2 + (\delta y)^2} < (\delta x + \delta y)\,.$$

In this case,

$$\delta q = \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{(0.05)^2 + (0.03)^2} = 0.06\,.$$

You would then report the final answer for q as

$$q = 11.07 \pm 0.06\, m\ .$$

If you had not added the uncertainties in quadrature, you would have calculated the uncertainty

$$0.03 + 0.05 = 0.08$$

instead of the final uncertainty of 0.06.

Propagation of uncertainties in products and quotients

Addition in quadrature is also used when calculating the final uncertainty δq of a quantity q composed of a product xy or quotient x/y. If q is a product or quotient of these measurements, you can find the final uncertainty δq by adding the fractional uncertainties of x and y. The fractional uncertainty of a quantity x is the ratio of δx to $|x|$ where $|x|$ is the absolute value of that quantity. The fractional uncertainty is written as

$$\text{Fractional uncertainty} = \frac{\delta x}{|x|}.$$

You should add these fractional uncertainties in quadrature to avoid a worst-case scenario, as stated in the previous example. When the fractional uncertainties are added in quadrature, it will yield the fractional uncertainty in the final quantity q. The fractional uncertainty in q is then written as

$$\frac{\delta q}{|q|} = \sqrt{\left(\frac{\delta x}{|x|}\right)^2 + \left(\frac{\delta y}{|y|}\right)^2}\ .$$

Consider an example of how to calculate $\delta q/|q|$ when the quantity q is the quotient of x and y,

$$q = \frac{x}{y}.$$

In this example, we will use the same values for x and y as were used in the previous example, where $x = 7.65 \pm 0.05\ m$ and $y = 3.42 \pm 0.03\ m$. You calculate q easily to be

$$q = \frac{7.65}{3.42} = 2.24\ .$$

The fractional uncertainty in x is

$$\frac{\delta x}{|x|} = \frac{0.05}{7.65} = 0.0065\ ,$$

and the fractional uncertainty in y is

$$\frac{\delta y}{|y|} = \frac{0.03}{3.42} = 0.0087\ .$$

Adding these fractional uncertainties in quadrature gives

$$\frac{\delta q}{|q|} = \sqrt{\left(\frac{\delta x}{|x|}\right)^2 + \left(\frac{\delta y}{|y|}\right)^2} = \sqrt{(0.0065)^2 + (0.0087)^2} = 0.0109.$$

The next step is to multiply $\delta q/|q|$ by q to get δq

$$0.0109 \cdot 2.24 = 0.02\,.$$

You can then report the final answer q as

$$q = 2.24 \pm 0.02\,.$$

General formula for error propagation

You have seen methods of propagating error through sums, differences, products, and quotients. A general formula exists for propagating errors in a quantity, composed of any combination of variables. This general formula for error propagation involves calculating partial derivatives of a quantity composed of many variables.

Partial derivatives involve taking a derivative of a quantity of many variables with respect to only one of the variables at a time. For example, suppose you have a quantity f

$$f = x^2 + 2xy + \sin(y)\,.$$

Take the derivative of this function with respect to x by treating y as a constant.

$$\frac{d(f(x,y))}{dx} = 2x + 2y + 0\,.$$

If you were taking the partial derivative of this quantity with respect to y then, we treat x as a constant to get

$$\frac{d(f(x,y))}{dy} = 0 + 2y + \cos(y).$$

All the rules of derivation apply to partial derivatives. You have to be careful about which variable you are differentiating with respect to and treat the others as constants.

Armed with the concept of partial derivative, we now return to the general problem of error propagation. The general formula for error propagation is for a quantity q of several variables $x \ldots z$.

$$\delta q = \sqrt{\left(\frac{\partial q}{\partial x}\delta x\right)^2 + \left(\frac{\partial q}{\partial y}\delta y\right)^2 + \ldots + \left(\frac{\partial q}{\partial z}\delta z\right)^2}$$

This formula is for any number and any combination of variables.

Here is an example of using the general formula. Imagine you are to find the acceleration of gravity using a simple pendulum. You measure the length of the string and the period of the pendulum and calculate the acceleration of gravity using the expression

$$g = {4\pi^2 l}/{T^2}\ .$$

The quantity g is a combination of the measured values of l and T. You need to calculate the partial derivatives of g with respect to l and T as shown below.

$$\frac{\partial g}{\partial l} = {4\pi^2}/{T^2}$$

$$\frac{\partial g}{\partial T} = -{8\pi^2 l}/{T^3}$$

Suppose you measured $l = 42.62\ cm$ with an uncertainty $\delta l = 0.1 cm$ and $T = 1.72\ s$ with an uncertainty $\delta T = 0.006\ s$.

Combining these into the general formula yields

$$\delta q = \sqrt{\left(\frac{\partial g}{\partial l}\delta l\right)^2 + \left(\frac{\partial g}{\partial T}\delta T\right)^2}$$

$$= \sqrt{\left(\frac{4\pi^2}{T^2}\delta l\right)^2 + \left(\frac{-8\pi^2 l}{T^3}\delta T\right)^2}$$

$$= \sqrt{\left(\frac{4\pi^2}{(1.72)^2}(0.001)\right)^2 + \left(\frac{-8\pi^2(.4262)}{(1.72)^3}(0.006)\right)^2}$$

$$= \sqrt{0.000178 + 0.00157} = 4.2\ cm/s^2 = 0.04\ m/s^2\ .$$

So with the example measurements given, we can report g as,

$$g = (9.80 \pm 0.04)\ m/s^2.$$

The previous cases of uncertainty propagation in sums, quotients, products and differences are all special cases of the general formula used in this example.

Average and Standard Deviation

What is an average?

Depending on the experiments, you perform in labs you might be expected to take down 10 or even more measurements for the same experiment. You will notice that you end up with different data every time you perform the measurement. This is due to random error.

To find the average of a measurement, you simply add up all the results of each measurement, and then divide that number by how many measurements you have. For example, suppose you measure three different values:

4.2, 4.5, and 5.1 .

You are given three measurements ranging from a low value of 4.2 to a high value of 5.1. To find the average, add up the values given and divide by 3:

$$\frac{4.2+4.5+5.1}{3}=4.6\,.$$

What is the standard deviation?

The standard deviation is another method to estimate the final uncertainty when taking several measurements of the same quantity. The previous sections explained propagating errors taken from single measurements, to find a final error δx. Standard deviation is a statistical method to find a final error δx. When you take multiple measurements, your measured values tend to group around the average value. Each measurement differs from the average value by a certain amount. This difference is denoted in this paper as the Greek symbol *capital delta* (Δ). In your density example, the average was 4.6, therefore the differences from the average of each of the three density measurements are:

$$\Delta_1 = 4.6 - 4.2 = 0.4$$
$$\Delta_2 = 4.6 - 4.5 = 0.1$$
$$\Delta_3 = 4.6 - 5.1 = -0.5$$

The final uncertainties cannot be found by using the differences shown above. An average of the differences becomes

$$\text{Average of differences} = \frac{0.4+0.1+(-0.5)}{3}=0\,.$$

In any set of measurements, the average of the differences will always be zero. Taking the average of the differences is not helpful. The best way to sneak around this annoyance is to square all the differences

$$\Delta_1^2 = (0.4)^2 = 0.16$$
$$\Delta_2^2 = (0.1)^2 = 0.01$$
$$\Delta_3^2 = (-0.5)^2 = 0.25\ .$$

By squaring all these differences we obtain positive numbers. The average of these squared differences will always be nonzero.

$$\text{Average of the squared differences} = \frac{0.16 + 0.01 + 0.25}{3} = 0.14\ .$$

Since you originally squared these differences you now need to 'un-square' the average by taking the square root.

$$\text{Square root of Average} = \sqrt{0.14} = 0.374\ g/cm^3\ .$$

This square of the average just found, is called the standard deviation. The Greek letter sigma (σ) is used to denote standard deviation. The standard deviation of this example is then

$$\sigma = 0.4\ g/cm^3\ .$$

You would then report your measured value as

$$4.6 \pm 0.4\ g/cm^3\ .$$

From the explanation given above, a general formula can be derived.

$$\sigma = \text{Standard Deviation} = \sqrt{\frac{1}{N}\sum (x_i - \bar{x})^2}\ ,$$

where N is the number of measurements, x_i is each measured value, $\bar{x}$ is the average value and the Greek letter *capital sigma* (Σ) denotes the total sum of all individual measurements. For the density example, the relationship of this formula and the data can be clarified in Table 1.

Trial Number I	Measured Value x_i	Difference $\Delta_i = x_i - \bar{x}$	Difference Squared Δ_i^2
1	4.2	0.4	0.16
2	4.5	0.1	0.01
3	5.1	-0.5	0.25
Total:	$\sum x_i = 13.8$	$\sum \Delta_i = 0$	$\sum \Delta^2{}_i = 0.42$
Average = $\bar{x} = {}^{13.8}\!/\!{}_3 = 4.6$	Standard deviation = $\sigma = \sqrt{\frac{0.42}{3}} = 0.374$		

Table 1: An example of Standard Deviation

Using uncertainty to evaluate the reliability and quality of data

The previous sections explained what uncertainties are and how to calculate them. This section will now explain how to use your stated uncertainties to draw conclusions about the reliability and quality of your measured data.

Absolute error

If you report your measurement of a quantity as

$$x \pm \delta x \,,$$

the δx portion represents the absolute error for that measurement. For example, if your measured density is

$$\rho = 18.2 \pm 1.4 \; g/cm^3 \,,$$

the absolute error is 1.4 g/cm^3.

Relative error

Suppose you report your measurement of a quantity as

$$x \pm \delta x \,.$$

The relative error, percentage, or fraction of the measurement that is uncertain is given by the following equation:

$$\text{Fraction of measurement that is uncertain} = \frac{\delta x}{|x|} \;.$$

This ratio is called the relative error or the fractional uncertainty. It is called relative, because it compares the error to the actual measurement itself. It is called fractional because the result is a ratio that tells you how much of your measurement is uncertain.

The fractional uncertainty is also commonly expressed as a percentage by multiplying the quotient by 100.

$$\text{Percentage Uncertainty} = \frac{\delta x}{|x|} \cdot 100 = \%\ \mathit{Error}$$

For example, suppose you report a density measurement to be

$$\rho = 18.2 \pm 1.4\ g/cm^3\ .$$

This measurement has the fractional uncertainty

$$\frac{\delta}{|\rho|} = \frac{1.4}{18.2} = 0.08\ .$$

The number 0.08 does not tell us much. Multiply this number by 100 to give us the percentage

$$0.08 \cdot 100 = 8\%$$

The quality of measured data can be determined by evaluating its fractional or percentage uncertainty. A percentage uncertainty of about 10% may be an indication of crude measurements. Very careful measurements might have a fractional uncertainty of 1% or 2%. These percentages are not stated as concrete rules, but merely as guides to help you check the quality of your measured data. Your professor or lab assistant will probably tell you how careful your measurements need to be by telling you what percentage or fractional uncertainty is required in a specific experiment.

Accuracy

Many experiments in the freshmen physics labs require you to measure quantities of a known value such as the speed of light, electron charge, or the density of gold. The known values for these quantities can be compared to values you measure in the lab. The comparison of your measured value to an accepted value will tell you how accurate your measurements were. Accuracy is the discrepancy between your measured value and the known or accepted value. A large discrepancy between the measured value and accepted value indicates some inaccuracy in your measurement. A small discrepancy indicates an accurate measurement. For example, consider the measured density stated in the example of relative error

$$\rho = 18.2 \pm 1.4\ g/cm^3.$$

If you compared this density to an accepted density ($\rho_{accepted}$),

$$\rho_{accepted} = 19.00\ g/cm^3\ .$$

The discrepancy between these two values is found by the difference between your measured value and the accepted value of the density

$$\text{Discrepancy} = 19.0 - 18.2 = 0.8\ g/cm^3 .$$

The discrepancy value of 0.8 is within the stated uncertainty of 1.4 in your measured value of density. When the discrepancy is smaller than the error then you can have confidence in your measurement. When the discrepancy is larger than the error, you may need to check your measurement for mistakes. The question is "How much larger than the error can the discrepancy be?" Usually you can keep your measurement if the discrepancy is within the range of two uncertainties. The following chart (Figure 1) shows four examples of measurements and errors.

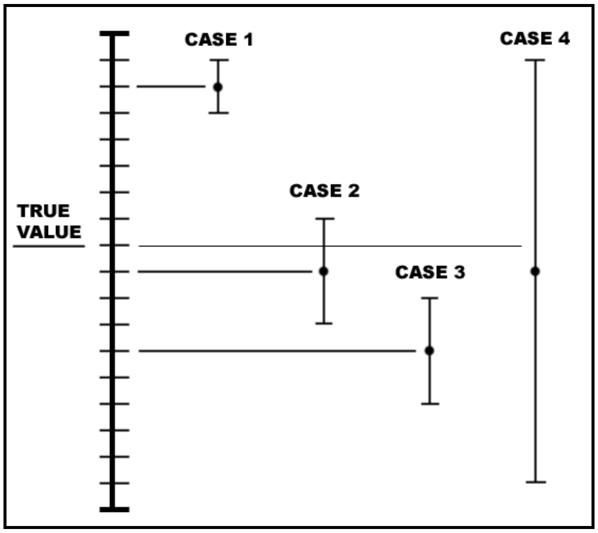

Figure 1: Four examples of error analysis
The dots are the measurements. The vertical bars are the uncertainty ranges for each measurement.

- Case 1: The true value is not close to the measured value. Since the true value lies far outside the range of the uncertainty, the measurement should be discarded.
- Case 2: This is an ideal measurement. The measurement is close to the accepted value, and the true value is well within the given uncertainty range.
- Case 3: This measurement is four units away from the accepted value, but since the true value is relatively close to the uncertainty, this may be valid. However, you may want to consult with your lab instructor who may have set guidelines about this case.
- Case 4: This measurement is certainly close to the true value, but the uncertainty is so large that it indicates a crude measurement. This large uncertainty prevents you from having confidence in the measurement. You may want to consult with your lab instructor who may have set guidelines about this case.

References:

Students can refer to other resources listed below.

A. Books:

Speigel, M. (2000) Schaum's Easy Outlines, Statistics. McGraw Hill.

You will enjoy this detailed, yet readable outline of the concepts of statistics. These concepts are useful for error analysis.

Taylor, John R. (1997) Introduction to Error Analysis, The Study of Uncertainties in Physical Measurements. University Science Books.

This is an advanced book on error analysis with lots of detail. You might want to familiarize yourself with this book. This is the required textbook for higher-level undergraduate physics laboratories.

B. Online Resources:

www.nilesonline.com/stats/ (visited 2002)

This site gives you detailed explanations on concepts such as Standard Deviation and data analysis. However, this site may be a bit harder to grasp if you have never been exposed to these concepts before.

http://www.lhup.edu/~dsimanek/errors.htm (visited 2002)

This is a good site on error analysis and no calculus background is required.

http://learn.chem.vt.edu/tutorials/error/ (visited 2002)

This site provides an index of the concepts in error analysis. You should be familiar with some mathematical notation.