Problem 1. The solution is given by the "starred" equation of HW 1's problem 3:

\[ E^2 = \frac{q}{4\pi\varepsilon_0} \left\{ \frac{(1 - \frac{v^2}{c^2})(R - \frac{v}{c}R)}{(R - \frac{v}{c}R)^3} + \frac{\vec{R}}{c^2} \times \frac{\left[(R - \frac{v}{c}R) \times \vec{a}\right]}{(R - \frac{v}{c}R)^3} \right\} \] (1)

Let us choose the xy-coordinate plane such that at the moment of time \( t' \), the charged particle is at the top of the circular orbit.

At \( t' \), particle was at the top, particle is now here \( t \)

\[ \vec{R} = -R \hat{y} \] (2)

The length of \( AB \) is

\[ \hat{A}B = \frac{R}{c} \]

\[ \vec{a} = -\frac{v^2}{R} \hat{y} \]

\[ \vec{v} = v \hat{x} \] (3)

Substituting Eqs. (2) - (3) into Eq. (1) we get:

\[ E_0^2 = \frac{q}{4\pi\varepsilon_0} \left\{ (1 - \beta^2) \left(-R \hat{y} - R \beta \hat{x}\right) + \frac{R(-\hat{y})}{c^2} \times \left[-\beta R \hat{x} \times \left(-\frac{v^2}{R} \hat{y}\right)\right] \right\} \]

Using \( \hat{y} \times (\hat{x} \times \hat{y}) = \hat{y} \times \hat{z} = \hat{x} \) we find:
\[ \mathbf{E}_0 = \frac{q}{4\pi \varepsilon_0 R^2} \left\{ -(1-\beta^2)(\beta \hat{x} + \hat{y}) - \beta^3 \hat{z} \right\} = \]
\[ = - \frac{q}{4\pi \varepsilon_0 R^2} \left( \beta \hat{x} + (1-\beta^2) \hat{y} \right) \]

In the non-relativistic case \( \beta \ll 1 \) we recover the usual Coulomb law
\[ \mathbf{E}_0 = -\frac{q \hat{r}}{4\pi \varepsilon_0 R^2} \]

In the ultra-relativistic limit, \( \beta \rightarrow 1 \)
\[ \mathbf{E}_0 = -\frac{q \hat{x}}{4\pi \varepsilon_0 R^2}, \] the same value but "rotated" by 90°.

Note that the maximum value of \( |\mathbf{E}| \) is achieved when
\[ \frac{d}{d\beta} \left( \beta^2 + (1-\beta^2)^2 \right) = 0 \quad \Rightarrow \quad 2\beta - 4\beta(1-\beta^2) = 0 \]
\[ \Rightarrow \beta = \frac{1}{\sqrt{2}} \]

Magnetic field \( \mathbf{B} = \frac{\mathbf{R} \times \mathbf{E}}{c} = \]
\[ = -\frac{\mu_0 q \hat{v}}{4\pi R^2} \hat{z} \text{ for all velocities } \mathbf{v}. \]
Problem 2

Vector potential is given by

\[ \mathbf{A}(\mathbf{r}, t) = \mathcal{Z} \int_{\mathcal{A}} d\mathbf{r} \frac{\mathbf{j}(\mathbf{r}, t - \frac{\mathbf{R} - \mathbf{r}}{c})}{4\pi |\mathbf{R} - \mathbf{r}|} \]

The current (along z-direction) is

\[ \mathbf{j}(\mathbf{r}, t) = \dot{q}(t) \mathbf{\hat{z}} \Theta(\frac{R}{2} - 121) \]

where the \( \Theta \)-function takes into account the finite length of the dipole.

\[ \mathbf{A}(\mathbf{r}, t) = \mathcal{Z} \mu_0 \int d\mathbf{z} \frac{\dot{q}(t - \frac{\mathbf{R} - \mathbf{z}}{c})}{4\pi |\mathbf{R} - \mathbf{z}|} \]

The condition \( R >> l \) allows to disregard \( \mathbf{\hat{z}} \) in comparison with \( \mathbf{R} \):

\[ \mathbf{A}(\mathbf{r}, t) = \mathcal{Z} \mu_0 \frac{\dot{q}(t - \frac{\mathbf{R}}{c}) l}{4\pi R} = \mu_0 \frac{\ddot{q}(t - \frac{\mathbf{R}}{c})}{4\pi R} \]

where \( \ddot{\mathbf{d}} = \mathcal{Z} q \mathbf{l} \) is the electric dipole moment. Magnetic field now

\[ \mathbf{B} = \nabla \times \mathbf{A} = \mu_0 \nabla \times \frac{\ddot{q}(t - \frac{\mathbf{R}}{c})}{4\pi R} \]

The condition (far field) \( R >> \frac{c}{\omega} \) ensures that the leading term comes from differentiating the dipole moment.
\[ B = \frac{1}{4\pi R} \nabla \times \mathbf{d}(t - \frac{R}{c}) = \frac{M_0}{4\pi R} \mathbf{d}(t - \frac{R}{c}) \times \nabla \frac{R}{c} = \]
\[ = \frac{M_0}{4\pi c R} \mathbf{d}(t - \frac{R}{c}) \times \hat{R} \]

In the far field region \( \mathbf{E} = c \mathbf{B} \times \hat{R} \)
(can you prove it using both \( \mathbf{A} \) and a solution for \( \Phi \) - found similarly to the above?)

\[ \mathbf{E} = \frac{M_0}{4\pi R} (\mathbf{d} \times \hat{R}) \times \hat{R} \]

\[ \mathbf{B} \rightarrow \mathbf{S} = \frac{\mathbf{E} \times \mathbf{B}}{M_0} = \frac{c \mathbf{B}^2}{M_0} \hat{R} \]
\[ = \frac{M_0 (\mathbf{d} \times \hat{R})^2}{16\pi^2 c R^2} \]

Radiation intensity:

\[ \frac{dI}{d\omega} = |\mathbf{S}| R^2 = \frac{M_0}{16\pi^2 c} (\mathbf{d} \times \hat{R})^2 = \frac{M_0 \hat{d}^2 \hat{R}^2 \sin^2 \Theta}{16\pi^2 c} \]
Problem 3 The "starred" equation on page 8 of the last problem's (Homework solution) gives electric field of a moving point charge at an arbitrary distance from it. The far field part of it, \( \sim 1/R \), is given by the second term:

\[
\mathbf{E} = \frac{q}{4\pi \varepsilon_0 c^2} \mathbf{R} \times \frac{(\mathbf{R} - \frac{\mathbf{v}}{c} \cdot \mathbf{R}) \times \mathbf{a}}{(\mathbf{R} - \frac{\mathbf{v}}{c} \cdot \mathbf{R})^3}
\]

Energy flux is given by \( \mathbf{S} = \frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \), since \( \mathbf{B} = \frac{\mathbf{R} \times \mathbf{E}}{c} \), where \( \mathbf{R} = \frac{\mathbf{R}}{R} \).

The radiation intensity

\[
\frac{dI}{d\Omega} = |\mathbf{S}| R^2 = \frac{q^2}{16 \pi \varepsilon_0 c^3} \frac{R^2}{(1 - \frac{\mathbf{v} \cdot \mathbf{R}}{c})^6}
\]

where we denoted

\[
\mathbf{K} = \mathbf{R} \times \left[ (\mathbf{R} - \frac{\mathbf{v}}{c} \cdot \mathbf{R}) \times \mathbf{a} \right]
\]

\[
\mathbf{K} = (\mathbf{R} - \frac{\mathbf{v}}{c} \cdot \mathbf{R}) (\mathbf{R} \cdot \mathbf{a}) - \mathbf{a} \left( 1 - \frac{\mathbf{v} \cdot \mathbf{R}}{c} \right)
\]

Let us choose spherical angles as shown in the picture (next page)
Substitution yields,

\[ \vec{K} = a \sin \theta \cos \phi \left( x \sin \theta \cos \phi + y \sin \theta \sin \phi + z (\cos \theta - \beta) \right) - a \vec{x} (1 - \beta \cos \theta) \]

Calculating now \( \vec{K}^2 \),

\[ \vec{K}^2 = a^2 \left[ (\sin^2 \theta \cos^2 \phi - 1 + \beta \cos \theta)^2 + \sin^2 \theta \cos^2 \phi + \sin^2 \phi (\cos \theta - \beta)^2 \right] \]

after straightforward simplifications we find

\[ \vec{K}^2 = a^2 \left[ (1 - \beta \cos \theta)^2 - \sin^2 \phi \cos^2 \phi (1 - \beta^2) \right], \]

which yields the intensity \( \left( \frac{1}{\varepsilon_0 c^3} = \frac{M_0}{c^3} \right) \):

\[ \frac{dI}{d\vec{R}} = \frac{\mu_0 q^2}{16\pi^2 c^6} \frac{(1 - \beta \cos \theta)^2 - \sin^2 \phi \cos^2 \phi (1 - \beta^2)}{(1 - \beta \cos \theta)^6} \] (1)

We observe that the maximum of \( \frac{dI}{d\vec{R}} \) is always achieved for \( \phi = \pi/2 \) when the second term in Eq. (1) is zero.
In the ultra-relativistic limit of $\beta \to 1$
small angles of $\theta \approx 0$ feature large intensity as
the denominator can become very small:

$$1 - \beta \cos \theta \approx 1 - \beta (1 - \frac{\theta^2}{2}) \approx 1 - \beta + \frac{\theta^2}{2} = \zeta + \frac{\theta^2}{2}$$

where $\zeta = 1 - \beta \ll 1$

(notice how in the small $\theta^2$ term it is
sufficient to set $\beta = 1$)

$$\sin^2 \theta \cos^2 \phi (1 - \beta^2) = 2 \theta^2 (1 - \beta) \cos^2 \phi = 2 \zeta \theta^2 \cos^2 \phi$$

Thus, the second term in the numerator of
Eq. (1) is small, compared with the first one

$$2 \zeta \theta^2 \cos^2 \phi \ll \zeta + \frac{\theta^2}{2}$$

and can be neglected:

$$\frac{dI}{d\Omega} \approx \frac{\mu g v^2}{16 \pi^2} \frac{1}{(\zeta + \theta^2)^4}$$

The intensity is progressively more peaked
function when $\zeta \to 0$, most of the radiation
is within the interval of angles $\theta \approx \sqrt{\zeta}$
in the forward direction (along $\vec{v}$).