Problem 1

Probability to have $N_s$ photons in a state $S = (k, z)$, where $z = \pm 1$ (for two polarizations), is

$$w_s(N_s) = e^{-\frac{\hbar w_s}{k_B T}(N_s + \frac{1}{2})}$$

Partition function is found from normalization condition:

$$Z = \sum_{N_s=0}^{\infty} e^{-\frac{\hbar w_s}{k_B T}(N_s + \frac{1}{2})} = \frac{e^{-\frac{x}{2}}}{1 - e^{-x}}, \text{ where } x = \frac{\hbar w_s}{k_B T}$$

Thus,

$$w_s(N_s) = e^{-N_s x} (1 - e^{-x})$$

Average number of photons in the state's is $N_s$ is

$$\bar{N_s} = \sum_{N_s=0}^{\infty} N_s w_s(N_s) = (1 - e^{-x}) \sum_{N_s=0}^{\infty} N_s e^{-N_s x} =$$

$$= -(1 - e^{-x}) \frac{\partial}{\partial x} \sum_{N_s=0}^{\infty} e^{-N_s x} = -(1 - e^{-x}) \frac{\partial}{\partial x} \frac{1}{1 - e^{-x}} =$$

$$= (1 - e^{-x}) \frac{e^{-x}}{(1 - e^{-x})^2} = \frac{1}{e - 1} = \frac{\hbar w_s}{k_B T} - 1$$

This is Bose-Einstein distribution with zero chemical potential.

Average energy in state $s$ is $E_s = \hbar w_s \bar{N_s}$, where $w_s = c/k$.
The total energy (per unit volume) is found by multiplying by the number of states per element of k-space:

\[
\frac{E}{V} = 2 \int \frac{d^3 k}{(2\pi)^3} \frac{\hbar \omega}{e^{\frac{\hbar \omega}{kBT}} - 1}
\]

To account for polarization degeneracy, using \( d^3 k = 4\pi k^2 dk = 4\pi \omega^2 d\omega / c^3 \), we write:

\[
\frac{E}{V} = \frac{\hbar}{\pi^2 c^3} \int_0^\infty \frac{\omega^3 d\omega}{e^{\frac{\hbar \omega}{kBT}} - 1}
\]

This can be re-written as \( \frac{E}{V} = \int_0^\infty \rho(\omega) d\omega \) where spectral energy density

\[
\rho(\omega) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\frac{\hbar \omega}{kBT}} - 1}
\]

has the form

The of \( \rho(\omega) \) is found from

\[
\left( \frac{x^3}{e^{x}} \right) = \frac{3x^2(e^{-x}) - xe^{-x}}{e^{-x}} = 0 \Rightarrow 3(e^{-x}) = xe^{-x}
\]

\[
x = 2.62 \quad \Rightarrow \quad \omega = \frac{kBT}{\hbar} \times 2.82
\]
Problem 2

\[ \Psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} \]

\[ \Psi_m(x) = \sqrt{\frac{2}{L}} \sin \frac{m\pi x}{L} \]

\[ d_{nm} = \frac{2q}{L} \int_0^L dx x \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L}, \quad n \neq m \]

Using \( \sin a \sin b = \frac{1}{2} \left[ \cos(a-b) - \cos(a+b) \right] \)

we can write

\[ d_{nm} = \frac{q}{L} \int_0^L dx x \left[ \cos \frac{n(n-m)x}{L} - \cos \frac{n(n+m)x}{L} \right] \]

the first integral:

\[ \frac{1}{L} \int_0^L dx x \cos \frac{n(n-m)x}{L} = \frac{x}{n(n-m)} \sin \frac{n(n-m)x}{L} \bigg|_0^L \]

\[ - \frac{1}{n(n-m)} \int_0^L dx \sin \frac{n(n-m)x}{L} = \]

\[ = \frac{L}{n^2(n-m)^2} \cos \frac{n(n-m)x}{L} \bigg|_0^L = -\frac{L}{n^2(n-m)^2} (1 - (-1)^{n-m}) \]

Similarly, the second integral in Eq. (1) is obtained by replacing \( m \to -m \):

\[ d_{nm} = \frac{q}{n^2} \left[ \frac{1 - (-1)^{n+m}}{(n+m)^2} - \frac{1 - (-1)^{n-m}}{(n-m)^2} \right] \]

This expression vanishes if \( \Delta n = n - m \) is even (\( n+m \) is then even too) — only transitions between states of opposite parity are allowed, since operator \( \hat{\tau} = \frac{q}{\hbar} \hat{\tau} \) is odd under spatial inversion (with respect to \( x = L/2 \)).
Problem 3

The rate of transition is given by the equation (see Feb. 12 lecture notes)

\[ w = \frac{\omega^3}{3 \pi c^3 \varepsilon_0 \hbar}, \quad \text{where} \quad \hbar w = \frac{m e^4}{2 (4 \pi \varepsilon_0)^{3/2} \hbar} \left( 1 - \frac{1}{4} \right) = \frac{3 m e^4}{8 (4 \pi \varepsilon_0 \hbar)^2} \]

Only the operator has non-zero matrix element for the \( 2p \rightarrow 1s \) transition

\[ \langle 2p \rangle_f = e \int r^2 dr d\sin \theta d\phi \cdot r \cos \theta \psi_2p(r, \theta) \psi_1s(r) \]

\[ = \frac{2 \pi e}{\sqrt{4\pi}} \frac{2}{\sqrt{3}} \int_0^\infty \frac{3^{3/2}}{2^{1/2} 3^{1/2}} \sqrt{\frac{1}{4\pi}} \sqrt{\frac{3}{4\pi}} \int_0^\infty r^2 dr (\frac{r}{a_B}) e^{-\frac{3r}{2a_B}} \int_0^{\pi/2} \sin^2 \theta \cos \theta d\theta \]

The integrals here

\[ \int_0^\infty r^2 dr (\frac{r}{a_B}) e^{-\frac{3r}{2a_B}} = a_B^3 \int_0^\infty r^3 e^{-\frac{3r}{2}} = \frac{256}{81} a_B^4 \]

\[ \int_0^{\pi/2} \sin^2 \theta \cos \theta d\theta = \int_0^1 dy y^2 = \frac{2}{3} \]
All together:

\[
\langle dt \rangle_f = \frac{2\pi e a_B}{\sqrt{2} 4\pi} \frac{256}{81} \times \frac{2}{3} = \frac{128\sqrt{2}}{243} e a_B
\]

All together:

\[
w = \frac{1}{3\pi c^3 \varepsilon_0 h} \left( \frac{3}{8} \frac{m e^4}{(4\pi \varepsilon_0 h)^2} \right)^3 \left( \frac{128\sqrt{2}}{243} \right)^2 e a_B^2
\]

\[
= \frac{k^4}{m a_B^2}
\]

also \( \frac{e^2}{4\pi \varepsilon_0 c h} = \alpha = \frac{1}{137} \)

\[
w = \frac{4}{3} \frac{\alpha a_B^2}{c^2} \left( \frac{3}{8} \right)^3 \frac{h^3}{m^3 a_B^6} \left( \frac{128\sqrt{2}}{243} \right)^2
\]

\[
= \frac{256}{6561} \alpha \frac{h^3}{c^2 m^3 a_B^4}
\]