Feynman's derivation of the Schrödinger equation[a)]

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R. P. Feynman's "path integral" approach to quantum field theory emerged from his discovery that the Schrödinger equation could be derived from an expression of P. A. M. Dirac's involving the action. Feynman gave the history of his discovery in his Nobel lecture, but withheld most of the details; when he did publish the discovery, he did not provide the framework. This article provides some historical background and puts both of Feynman's presentations together.

I. INTRODUCTION

A recent book by Schweber[1] describes the early history of quantum electrodynamics, the first example of quantum field theory. If a student picks up Schweber's book, or any text of quantum field theory, she will find the name of Feynman writ very large. Many of the standard techniques of quantum field theory derive from Feynman's "sum over histories" approach to quantum mechanics. Feynman's first triumph with this approach was his derivation of the Schrödinger equation. Feynman, a consummate teacher with a unique gift for simplicity, presented his work to two very different audiences. In his Nobel lecture,[2] he explained how he came to the calculation, but left out most of the mathematics. For the experts he worked through it more than once,[3] but did not provide much of a starting point. This article attempts to put together these two descriptions into a story more accessible than either alone.

II. FEYNMAN'S BACKGROUND: NUMERICS AND LAGRANGIANS

From an early age Feynman taught himself as much mathematics as he could, not only theorems but the practical and numerical parts as well.[4] His calculational skill was legendary, even among those whose careers were devoted to numerical analysis. For example, in Acton[5] we read:

> Art, mostly in the form of intuition, still plays too large a part to permit consistent results when invoked by different persons... In the hands of a Feynman the technique works like a Latin charm; with ordinary mortals the result is a mixed bag.

Consequently, given any formal expression, Feynman was very likely to play around with it, and as far as possible, winkle numbers out of the thing. In particular, he was a master of approximation.[6] It is said that at Los Alamos, Enrico Fermi, John von Neumann and Feynman would check each other's calculations, Fermi using a slide rule, Feynman a Marchant (mechanical) calculator, and von Neumann his head; all three obtaining very nearly the same answer at very nearly the same time.

Feynman seems to have had unusually good high school teachers. One of these, a Mr. Bader, introduced Feynman to the Euler–Lagrange equations, and Lagrange's approach to mechanics. Feynman's account of this is fascinating.[7] This material is taught usually in the second year of an American physics degree.

In 1788 Joseph Louis Lagrange, developing earlier work of Leonhard Euler, published *Méchanique Analytique*, a new approach to Newtonian mechanics based on the calculus of variations. A familiar problem in calculus is to find the point at which a given function takes on its greatest or least value. The corresponding problem in the calculus of variations is, given the integral of an unknown function, find the function that makes the value of the integral a maximum or minimum. The calculus of variations, one of Feynman's favorite and most powerful techniques,[8] arose from a single problem, the *brachistochrone*:[9] Given two points A and B in a vertical plane, find the shape of a ramp which will take a mass sliding under the force of gravity from A to B in the shortest possible time. The problem was originally published in 1696 by Johann Bernoulli, who apparently hoped that his brother, Jakob, would fail to solve it. Though his solution took several months (Newton is said to have solved it the same night he read it!), Jakob alone recognized the essential point: any curve which has a minimum property globally (in the large) must have that same property *locally* (in the small), as Bernoulli's diagram (see Fig. 1) will help make clear:

Suppose the curve ACEDB has the desired minimal property (in this case, the path of least time from A to B). Let C and D be two points on the curve. Then, said Bernoulli, CED must have this same property, i.e., it must be the quickest path from C to D. If it were not, then there would be some other path, CFD say, which would be faster from C to D. But, were that so, the path ACFDB would be faster than the path ACEDB, which is contrary to hypothesis. Thus, the path quickest overall must be quickest in between any intermediary points, and the property which holds globally must also hold locally.

Bernoulli's "local principle" turns an integral equation (the time from A to B) into a differential equation, far more amenable to analysis. As will be seen, this is precisely what Feynman did.

Lagrange's approach concerns the integral of the difference of kinetic and potential energies. Denote the kinetic energy by K and the potential energy by U. The kinetic energy K depends on the object's velocity $v=dx/dt$, while the potential energy depends usually only on the object's position x. Then, form the Lagrangian function L as the difference $K-U$. Now, said Lagrange, insist that the actual physical path followed by an object is just that unique trajectory which minimizes the integral S;

$$S=\int_{t_1}^{t_2} L\,dt=\int_{t_1}^{t_2}(K-U)\,dt. \tag{2.1}$$

In the past, this integral was called "Hamilton's Principal Function" after the 19th century Irish mathematician Will-

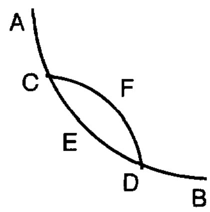

Fig. 1. Bernoulli's diagram.

iam Rowan Hamilton. Today, following Feynman, everyone calls it "the action." The methods of the calculus of variations applied to the action S lead to the Euler–Lagrange equation

$$\frac{\partial L}{\partial x}-\frac{d}{dt}\left(\frac{\partial L}{\partial(dx/dt)}\right)=0. \tag{2.2}$$

In the simplest cases, $K=\frac{1}{2}mv^2$ where m is a (constant) mass and U is a function of x only, so the Euler–Lagrange equation produces

$$-\frac{\partial U}{\partial x}=\frac{d}{dt}(mv), \tag{2.3}$$

which is just Newton's second law. That is, the path that minimizes the action S is the *same* path for which Newton's second law holds true. Since the second law works well to describe the paths of objects, it seems to be the case that somehow nature chooses, out of an infinite set of paths, the *one* that minimizes S.

III. FROM CLASSICAL TO QUANTUM MECHANICS

The fundamental equation for quantum mechanics, analogous to Newton's second law for classical mechanics, is Schrödinger's equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+U\psi=i\hbar\frac{\partial\psi}{\partial t}. \tag{3.1}$$

Schrödinger obtained this equation by analogy with the connection between geometric optics, when light is treated as if it consisted of small particles, and wave optics, in which the wave nature of light determines its behavior. In geometric optics, the path of light may be found also with a variational principle, Fermat's principle: Light travels from one given point to another in the shortest time. Mathematically, this means that the phase ϕ of the light wave must vary as little as possible from one point to another. The wave Z itself may be described by a function of the form[10]

$$Z=ae^{i\phi}, \tag{3.2}$$

where Z is the wave function, a its amplitude and ϕ its phase. If we ignore the wave nature of light, treating the light as particles, we effectively take the wave length to be very small. For classical mechanics, we minimize the action S, just as in Fermat's principle we minimize ϕ. So a reasonable thing to do is to *guess* with Schrödinger that the light wave phase ϕ corresponds to the action S, and consequently a wave description of mechanics *might* take the approximate form[11]

$$\psi=ae^{iS/\hbar}. \tag{3.3}$$

The division by Planck's constant is to make the argument of the exponential dimensionless; both Planck's constant and S have the same dimensions. Schrödinger knew that the action S obeyed the Hamilton–Jacobi equation;

$$\frac{\partial S}{\partial t}+\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2+U=0. \tag{3.4}$$

(In many cases, this is just the conservation of energy in disguise; the first term equals $-E$, and the second term is the square of the momentum mv divided by $2m$, or the kinetic energy.) Using the guess for ψ above, it follows:

$$\frac{\partial S}{\partial x}=-\frac{i\hbar}{\psi}\frac{\partial\psi}{\partial x} \tag{3.5}$$

and making due allowance for the complex nature of ψ, we can recast the Hamilton–Jacobi equation as

$$-E+\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^*\left(\frac{\partial S}{\partial x}\right)+U=0, \tag{3.6}$$

where the * indicates complex conjugation. That is,

$$-E+\frac{1}{2m}\left(\frac{i\hbar}{\psi^*}\frac{\partial\psi^*}{\partial x}\right)\left(-\frac{i\hbar}{\psi}\frac{\partial\psi}{\partial x}\right)+U=0 \tag{3.7}$$

or rearranging

$$(U-E)\psi^*\psi+\frac{\hbar^2}{2m}\left(\frac{\partial\psi^*}{\partial x}\right)\left(\frac{\partial\psi}{\partial x}\right)=0 \tag{3.8}$$

We can call the expression on the left hand side M and, following Schrödinger, use it as a Lagrangian with generalized coordinates ψ, ψ^*, $\partial\psi/\partial x$, and $\partial\psi^*/\partial x$. That is, we minimize the integral T;

$$T=\int M\,dx=\int (U-E)\psi^*\psi+\frac{\hbar^2}{2m}\left(\frac{\partial\psi^*}{\partial x}\right)\left(\frac{\partial\psi}{\partial x}\right)dx. \tag{3.9}$$

The Euler–Lagrange equation (for ψ^*) then gives

$$\frac{\partial M}{\partial\psi^*}-\frac{\partial}{\partial x}\left(\frac{\partial M}{\partial(\partial\psi^*/\partial x)}\right)=0 \tag{3.10}$$

or

$$(U-E)\psi-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}=0, \tag{3.11}$$

which, after rearrangement, is just the (time-independent) Schrödinger equation.

IV. FEYNMAN'S DERIVATION

Feynman's Nobel lecture describes his earliest attempts while a graduate student of John A. Wheeler's at Princeton to make sense of the quantum theory of photons and electrons. He soon became convinced that the Lagrangian was the key to the problem. While attending a party at the Nassau Tavern, Feynman asked a visiting European professor, Herbert Jehle, if he knew of any work incorporating the Lagrangian into quantum mechanics. Jehle did, and the next day showed Feynman an article by P. A. M. Dirac.[12]

To describe Dirac's work, it may be helpful to recall another analogy from optics, Huygens' principle, which treats light not as particles but as waves. Given one wave front, Huygens took that front as the progenitor of smaller waves, which collectively made up the next wave. In this manner the wave disturbance propagated forwards. Mathematically, Huygens' principle takes the form of an integral equation:

$$\psi(x,t_2)=\int G(x,y)\psi(y,t_1)dy \quad \text{where } t_2>t_1. \tag{4.1}$$

The function $G(x,y)$ is called a "kernel" or a "propagator." Dirac's article made the suggestion that this kernel was "analogous" to an expression we have already seen, namely $\exp(iS/\hbar)$. It would be criminal not to let Feynman tell in his own words what came next:[13]

> Professor Jehle showed me this, I read it, he explained it to me, and I said, "what does he mean, they are analogous; what does that mean, *analogous*? What is the use of that?" He said, "you Americans! You always want to find a use for everything!" I said that I thought that Dirac must mean that they were equal. "No", he explained, "he doesn't mean they are equal." "Well," I said, "let's see what happens if we make them equal."

As Bernoulli before him, Feynman knew that if it was true for any two times, it had to be true for times separated by a small interval Δt, which he denoted ϵ. This would allow him to use all his approximation techniques. Then

$$S=\int L\,dt\approx L_{\text{average}}\epsilon. \tag{4.2}$$

In the simplest case, the Lagrangian is just the difference of the kinetic energy, $K=\frac{1}{2}mv^2$, and a potential energy $U(x)$. Then

$$K_{\text{average}}=(\tfrac{1}{2})m(\Delta x/\Delta t)^2=(\tfrac{1}{2})m(x-y)^2/\epsilon^2, \tag{4.3a}$$

$$U_{\text{average}}=U(\tfrac{1}{2}(x+y)), \tag{4.3b}$$

$$S\approx\tfrac{1}{2}m(x-y)^2\epsilon-U(\tfrac{1}{2}(x+y))\epsilon, \tag{4.3c}$$

so if Feynman's first guess was correct,

$$G(x,y)=\exp(iS/\hbar)$$
$$\approx\exp\{(\mathrm{im}(x-y)^2/2\epsilon\hbar)-(U(\tfrac{1}{2}(x+y))\epsilon/\hbar)\}. \tag{4.4}$$

The exponential of the sum is the product of the exponentials, so the kernel becomes

$$G(x,y)\approx\exp(\mathrm{im}(x-y)^2/2\epsilon\hbar)$$
$$\times\exp(-U(\tfrac{1}{2}(x+y))\epsilon/\hbar). \tag{4.5}$$

The first factor has ϵ in the denominator, and the second in the numerator. Since ϵ is supposed to be small, the second factor may be expanded in the well-known Taylor series for $\exp(x)$. That is, to first order in ϵ,

$$G(x,y)\approx\exp(\mathrm{im}(x-y)^2/2\epsilon\hbar)(1-U(\tfrac{1}{2}(x+y))\epsilon/\hbar). \tag{4.6}$$

Putting this expression into Dirac's integral equation gives

$$\psi(x,t+\epsilon)\approx\int \exp\left(\frac{\mathrm{im}(x-y)^2}{2\hbar\epsilon}\right)$$
$$\times\left\{1-\frac{i}{\hbar}\,\epsilon U\left(\frac{x+y}{2}\right)\right\}\psi(y,t)dy. \tag{4.7}$$

Although the integration extends from $-\infty$ to ∞, Feynman felt that the rapid oscillation of the exponential factor (due to the small sizes of Planck's constant and the time interval) would cause the integrand to be very small, except where $x-y$ was likewise small. So, he decided to rewrite the integral in terms of the difference $x-y=\xi$;

$$\psi(x,t+\epsilon)\approx\int \exp\left(\frac{\mathrm{im}\xi^2}{2\hbar\epsilon}\right)\left\{1-\frac{i}{\hbar}\,\epsilon U\left(x-\frac{1}{2}\,\xi\right)\right\}$$
$$\times\psi(x-\xi,t)d\xi. \tag{4.8}$$

Since the integrand is large only when ξ is small, it made sense to Feynman to expand $\psi(x-\xi)$ in a Taylor series:

$$\psi(x-\xi,t)=\psi(x,t)-\xi\,\frac{\partial\psi(x,t)}{\partial x}+\frac{1}{2}\,\xi^2\,\frac{\partial^2\psi(x,t)}{\partial x^2}-\cdots. \tag{4.9}$$

Evidently, it was at this point that Feynman discovered his first guess could not be exactly right, because to lowest order in ϵ the integral equation becomes

$$\psi(x,t)\approx\int \exp\left(\frac{\mathrm{im}\xi^2}{2\hbar\epsilon}\right)\psi(x,t)d\xi=\sqrt{\frac{2\pi i\hbar\epsilon}{m}}\,\psi(x,t) \tag{4.10}$$

from the well-known Gaussian integral, $\int\exp(-\alpha x^2)dx=\sqrt{\pi/\alpha}$. Consequently, the kernel G could not be equal to the expression $\exp(iS/\hbar)$; at best it could be only proportional. Feynman then put in a constant of proportionality A into his original guess [cf. (4.4)]:

$$G=A\,\exp(iS/\hbar) \quad \text{with } A=\sqrt{\frac{m}{2\pi i\hbar\epsilon}}. \tag{4.11}$$

Then, plugging in the Taylor expression, and writing $U(x-\xi/2)\,\epsilon=U(x)\epsilon$ to lowest order, the integral equation is

$$\psi(x,t+\epsilon)\approx A\int \exp\left(\frac{\mathrm{im}\xi^2}{2\hbar\epsilon}\right)\left\{1-\frac{i}{\hbar}\,\epsilon U(x)\right\}$$
$$\times\left\{\psi(x,t)-\xi\,\frac{\partial\psi}{\partial x}+\frac{1}{2}\,\xi^2\,\frac{\partial^2\psi}{\partial x^2}\right\}d\xi \tag{4.12}$$

$$\approx\left\{1-\frac{i}{\hbar}\,\epsilon U(x)\right\}\psi(x,t)$$
$$-A\,\frac{\partial\psi}{\partial x}\int \exp\left(\frac{\mathrm{im}\xi^2}{2\hbar\epsilon}\right)\xi d\xi$$
$$+\frac{1}{2}A\,\frac{\partial^2\psi}{\partial x^2}\int \exp\left(\frac{\mathrm{im}\xi^2}{2\hbar\epsilon}\right)\xi^2 d\xi. \tag{4.13}$$

(There are other terms, but these are the only ones which are no more than linear in the time interval, ϵ.) The term involving the first derivative of ψ is an odd function of ξ, so it equals zero. Then differentiating the Gauss integral above with respect to α (another technique Feynman was fond of[14]) yields the identity

$$\int x^2\exp(-\alpha x^2)dx=\frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}} \tag{4.14}$$

so that the term involving the second derivative of ψ is

$$\frac{1}{2}A\frac{\partial^2\psi}{\partial x^2}\int \exp\left(\frac{im\xi^2}{2\hbar\epsilon}\right)\xi^2\,d\xi$$

$$=\frac{1}{2}A\frac{\partial^2\psi}{\partial x^2}\frac{i\hbar\epsilon}{m}\sqrt{\frac{2\pi i\hbar\epsilon}{m}}=\frac{1}{2}\frac{i\hbar\epsilon}{m}\frac{\partial^2\psi}{\partial x^2}. \quad (4.15)$$

Gathering all these terms together, the integral equation becomes

$$\psi(x,t+\epsilon)\approx\psi(x,t)-\frac{i}{\hbar}\,\epsilon U(x)\psi(x,t)+\frac{1}{2}\frac{i\hbar\epsilon}{m}\frac{\partial^2\psi}{\partial x^2}, \quad (4.16)$$

which, after multiplying by $i\hbar$, can be rearranged into

$$i\hbar\,\frac{\psi(x,t+\epsilon)-\psi(x,t)}{\epsilon}\approx-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+U(x)\psi(x,t). \quad (4.17)$$

But the left-hand side is, to lowest order, just $i\hbar\partial\psi/\partial t$. This is *exactly* the (time-dependent) Schrödinger equation. Continuing with Feynman's Nobel lecture:

> So, I simply put them equal, taking the simplest example where the Lagrangian is $\frac{1}{2}M\dot{x}^2-V(x)$ but soon found I had to put a constant of proportionality A in, suitably adjusted. When I substituted $A\exp(i\epsilon L/\hbar)$ for [the kernel] to get [the equivalent of equation (3.7)], and just calculated things out by Taylor series expansion, out came the Schrödinger equation. So, I turned to Professor Jehle, not really understanding, and said, "well, you see Professor Dirac meant that they were proportional." Professor Jehle's eyes were bugging out—he had taken out a little notebook and was rapidly copying it down from the blackboard, and said, "no, no this is an important discovery. You Americans are always trying to find out how something can be used. That's a good way to discover things!" So, I thought I was finding out what Dirac meant, but, as a matter of fact, I had made the discovery that what Dirac thought was analogous was, in fact, equal. I had then, at least, the connection between the Lagrangian and quantum mechanics, but still with wave functions and infinitesimal times.

Within a day or two, Feynman realized that any arbitrary time Δt—equal to 2ϵ, 3ϵ, ..., $N\epsilon$—could be built up by this method, repeating the integration N-fold times. Such a procedure Feynman named "a sum over histories," and the resulting multiple integral a "path integral." Soon thereafter America entered the Second World War. Upon finishing his doctorate in the spring of 1942, Feynman set out for Los Alamos, where his energies would be entirely devoted to the construction of the atomic bomb. He was not to find the time to extend his methods fully to quantum electrodynamics until after the war's conclusion.

[a]In memory of Diana Van Valen.

[1]Silvan S. Schweber, *QED and the Men Who Made It* (Princeton U.P., Princeton, NJ, 1994).

[2]Richard P. Feynman, Nobel Lecture in Physics 1965, "The Development of the Space-Time View of Quantum Electrodynamics," Science **153**, 699–708 (1966); Reprinted in *Harvard Project Physics Reader*, "The Nucleus," edited by Gerald Holton, F. James Rutherford, and Fletcher G. Watson (Holt, Rinehart & Winston, New York 1970), Vol. 6.

[3]R. P. Feynman, "Space-Time Approach to Non-Relativistic Quantum Mechanics," Rev. Mod. Phys. **20**, 367–387 (1948), reprinted in J. Schwinger, *Quantum Electrodynamics* (Dover, New York 1958); R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965), Chap. 4. The bulk of the calculation in Sec. IV is taken directly from this source.

[4]James Gleick, *Genius: The Life and Science of Richard Feynman* (Pantheon, New York 1992), pp. 32–36; Ref. 1, p. 374.

[5]Forman S. Acton, *Numerical Methods That (usually) Work* (Mathematical Association of America reprint, 1990), p. 252. (The original was published by Harper and Row, New York 1972.)

[6]Reference 1, pp. 398–401; Reference 4 pp. 175–178; pp. 182–183. The lecture Gleick refers to here is probably not very different from Lecture 22, Vol. I, "Algebra," in R. P. Feynman, R. B. Leighton and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, MA, 1963); See also R. P. Feynman, *Surely You're Joking, Mr. Feynman* (Norton, New York, 1985), pp. 192–198, "Lucky Numbers."

[7]R. P. Feynman, R. B. Leighton and M. Sands, Ref. 6, Vol. II, Lecture 19, "The Principle of Least Action"; For more about Abraham Bader, see J. Mehra, *The Beat of a Different Drum: The Life and Science of Richard Feynman* (Oxford U.P., Oxford, UK, 1994), pp. 31–33.

[8]Reference 7; see also R. P. Feynman, *The Character of Physical Law* (Modern Library, New York 1994), pp. 46–52 and 97–99. (This is a reprint of the original MIT publication, Cambridge, MA, 1965.)

[9]Constance Reid, *Hilbert* (Springer-Verlag, Berlin, 1969), p. 68. For the original statement of the problem, as well as the solutions of both Bernoulli brothers, see D. J. Struik, *A Source Book in Mathematics, 1200–1800* (Harvard U.P., Cambridge, MA, 1969), pp. 391–99. (This book has been reprinted in paper covers by Princeton U.P., Princeton, NJ, 1993.)

[10]L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Addison-Wesley, Reading, MA, 1965), 2nd ed. p. 20. For the connection with the Hamilton-Jacobi equation, see p. 52.

[11]E. Schrödinger, "Quantisierung als Eigenwertproblem," Ann. Phys. **79**, 361–376 (1926). English translations of this paper and others in the same series may be found in *Wave Mechanics*, edited by Gunter Ludwig (Pergamon, Oxford, 1968), and E. Schrödinger, *Collected Papers on Wave Mechanics* (Chelsea, New York, 1978). Schrödinger did not at first make ψ a complex function, using instead the guess $S=K\ln\psi$ and ultimately requiring K to be equal to Planck's constant. Many physicists found Schrödinger's first derivation of his equation less than compelling; see, e.g., Walter Moore, *Schrödinger: Life and Thought* (Oxford U.P., Oxford, UK, 1989), p. 200, or for a contemporary account, G. Birtwistle, *The New Quantum Mechanics* (Cambridge U.P., Cambridge, UK, 1928), p. 137. I am here doing violence to history for the sake of pedagogy.

[12]P. A. M. Dirac, "The Lagrangian in Quantum Mechanics," Phys. Zeits. Sowjetunion **3** (1), 64–72 (1933). Reprinted in Schwinger, Ref. 3.

[13]R. P. Feynman, Nobel Lecture (Ref. 2).

[14]R. P. Feynman, *Surely You're Joking, Mr. Feynman*, pp. 86–87 (Ref. 6).