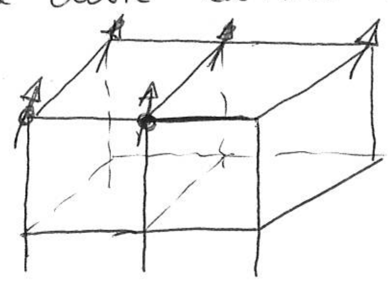
Ferromagnet

$$\hat{H} = -\frac{1}{2}\sum_{ij} \frac{J_{ij}}{\hbar^2} \vec{S}_i \vec{S}_{i+g}$$

$\hookrightarrow$ so that every bond contributes J_{ij} (and thus is counted once)

Simple cubic lattice : $J_{ij} = \begin{cases} J & \text{for } |i-j| = 1 \\ 0 & \text{otherwise} \end{cases}$



Every spin has z neighbors
$z=6$ for cubic lattice

Mean-field approximation (a.k.a. molecular-field approx):
replace neighboring spins by constant expectation value

$$\langle \vec{S}_{i+g} \rangle \rightarrow \langle S^z \rangle = \hbar \langle m_s \rangle$$

$$\hat{H} = -\sum_i \frac{zJ\langle m_s\rangle}{\hbar} S^z_i$$

$E = -zJ\langle m_s\rangle m_s$ for a given site i (which is arbitrary)

Then $$\langle m_{is} \rangle = \frac{\sum_{m_s=-S}^{S} m_s\, e^{\frac{zJ\langle m\rangle m_s}{k_BT}}}{\sum_{-S}^{S} e^{\frac{zJ\langle m\rangle m_s}{k_BT}}}$$

let $$x = \frac{zJ\langle m_s\rangle}{k_BT} = \frac{g\mu_B B_{mf}}{k_BT}$$

where $B_{mf} = \frac{zJ}{g\mu_B}\langle m_s\rangle$ is "molecular" (internal) field produced by spins surrounding a given one

Compare with calculation of magnetization (and susceptibility) for a single spin — it is exactly the same!

$$\langle m_s \rangle = \frac{\partial}{\partial x} \ln Z$$

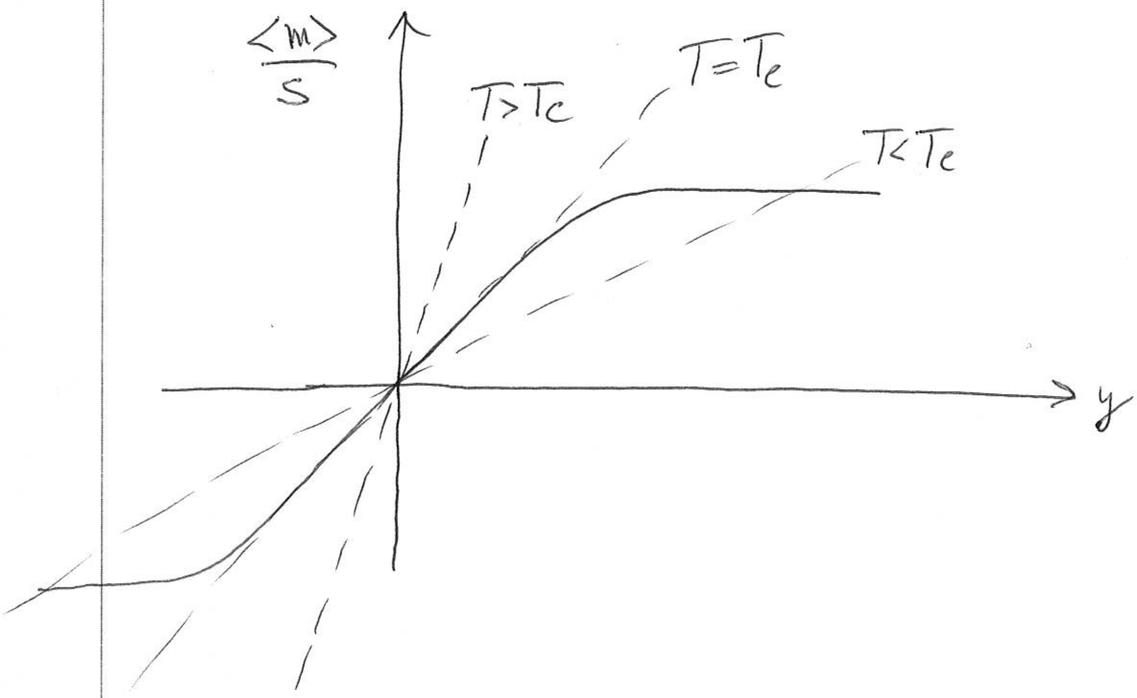
$$Z = \sum_{m=-S}^{S} e^{xm} = \frac{sh\left(\frac{2S+1}{2}x\right)}{sh\left(\frac{x}{2}\right)}$$

Thus $\langle m \rangle = S \left\{ \frac{2S+1}{2S} \coth\left(\frac{2S+1}{2S} \underbrace{\frac{zJS\langle m \rangle}{k_B T}} \right) - \frac{1}{2S} \coth\left(\frac{1}{2S} \frac{zJS\langle m \rangle}{k_B T} \right) \right\}$

previously denoted as $y = \frac{zJS\langle m \rangle}{k_B T}$

$$\frac{\langle m \rangle}{S} = B_S(y)$$

For $S = \frac{1}{2}$: $B_{\frac{1}{2}}(y) = \tanh(y) = th\left(\frac{zJ\langle m \rangle}{2 k_B T} \right)$



(3) $$Z = \sum_{m=-S}^{S} e^{xm} = \frac{\mathrm{sh}\left(\frac{2S+1}{2}x\right)}{\mathrm{sh}\left(\frac{x}{2}\right)}$$

$$\langle m \rangle = \frac{\partial \ln Z}{\partial x} = \left(S+\tfrac{1}{2}\right)\coth\left(S+\tfrac{1}{2}\right)x - \tfrac{1}{2}\coth\tfrac{1}{2}x$$

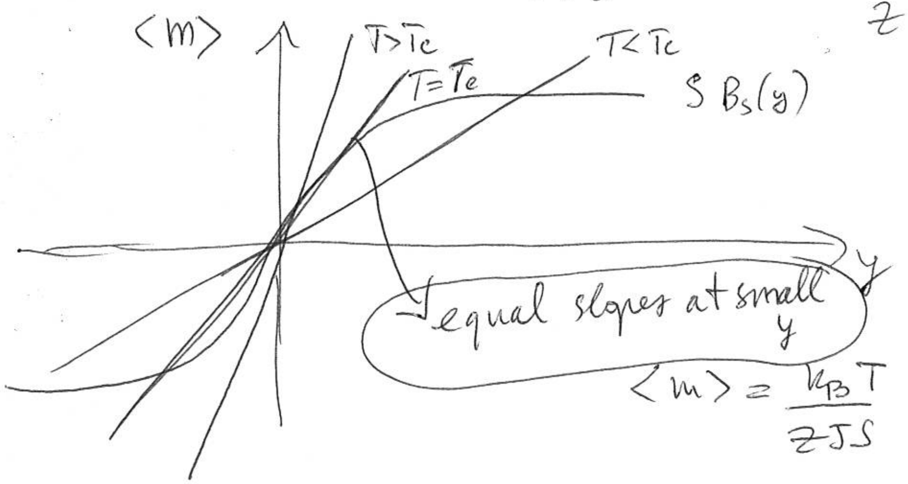
$$= (S)\left\{\frac{2S+1}{2S}\coth\left(\frac{2S+1}{2S}\frac{g\mu_B S B_{mf}}{k_B T}\right) - \frac{1}{2S}\coth\left(\frac{1}{2S}\frac{g\mu_B S B_{mf}}{k_B T}\right)\right\}$$

$$= \cancel{\langle S \rangle} S\, B_S(y) \quad \text{where} \begin{cases} y = \dfrac{g\mu_B S B_{mf}}{k_B T} \\ y = \dfrac{zJS\langle m\rangle}{k_B T} \end{cases}$$

$$\langle m \rangle = S\,\frac{S+1}{3S}\,y = \frac{S+1}{3}\,\frac{zJS\langle m\rangle}{k_B T_c}$$

$$T_c = \frac{zJS(S+1)}{3k_B}$$

$$M = \frac{N}{V} g\mu_B\langle m\rangle = \frac{N}{V}\frac{g^2\mu_B^2}{zJ} B_{mf}$$



$$\langle m \rangle = \frac{k_B T}{zJS}\, y$$

$T \approx T_c$

(4) $$B_S(y) = \frac{S+1}{3S} y - \xi y^3 = \frac{k_B T}{zJS^2} y$$

$$\xi y^3 = \left(\frac{S+1}{3S} - \frac{k_B T}{zJS^2}\right) y = \frac{1}{zJS^2}\left(\frac{zJS(S+1)}{3} - k_B T\right) y$$

$$= \frac{k_B y}{zJS^2}\left(T_c - T\right)$$

$$\Rightarrow \quad y^2 = \frac{k_B}{zJS^2} \frac{\left(T_c - T\right)}{\xi}$$

$$\frac{zJS}{k_B T} \langle m \rangle = \left(\frac{k_B (T_c - T)}{zJS^2 \xi}\right)^{1/2}$$

$$\xi = \frac{(2S+1)^4 - 1}{45 (2S)^4} \approx \frac{1}{45} \text{ for } \underline{2S \gg 1}$$

$$2S = 1 \qquad \frac{2^4 - 1}{45 \cdot 2^4} = \frac{15}{16 \cdot 45} = \frac{1}{48}$$

Thus, just below T_c, where small y expansion is justified

$$M \sim \sqrt{\frac{T_c - T}{T_c}} \qquad \text{square-root behavior}$$

$\underline{T \to 0} \Rightarrow y \to \infty$

Set $S = \frac{1}{2}$ for simplicity

$$B_{1/2}(y) = \tanh(y) = \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{1 - e^{-2y}}{1 + e^{-2y}} = 1 - 2e^{-2y} = 1 - 2e^{-2T_c/T}$$

here $\langle m \rangle = \frac{1}{2} \Rightarrow y = \frac{zJ}{4k_B T} \qquad T_c = \frac{zJ}{4k_B} \Rightarrow y = T_c/T$

Susceptibility above T_c

$$E = -zJ_g \langle m \rangle m_s \quad g\mu_B B m_s$$

$$\langle m \rangle = S\, B_S\left(\frac{zJS\langle m \rangle + g\mu_B BS}{k_B T}\right)$$

Above T_c $\quad B_S(y) \approx \frac{S+1}{3S}\, y$

$$\langle m \rangle \cong \frac{S+1}{3}\,\frac{zJS\langle m \rangle + g\mu_B BS}{k_B T} = \frac{zJS(S+1)}{3k_B T}\langle m \rangle + \frac{(S+1)g\mu_B BS}{3k_B T}$$

$$\langle m \rangle\left(1 - \frac{T_c}{T}\right) = \frac{(S+1)S g\mu_B B}{3k_B T}$$

$$\langle m \rangle = \frac{g\mu_B B(S+1)S}{3k_B(T - T_c)}$$

$$M = \frac{N}{V} g\mu_B \langle m \rangle = \frac{N}{V}\,\frac{g^2\mu_B^2 BS(S+1)}{3k_B}\,\frac{1}{T - T_c}$$

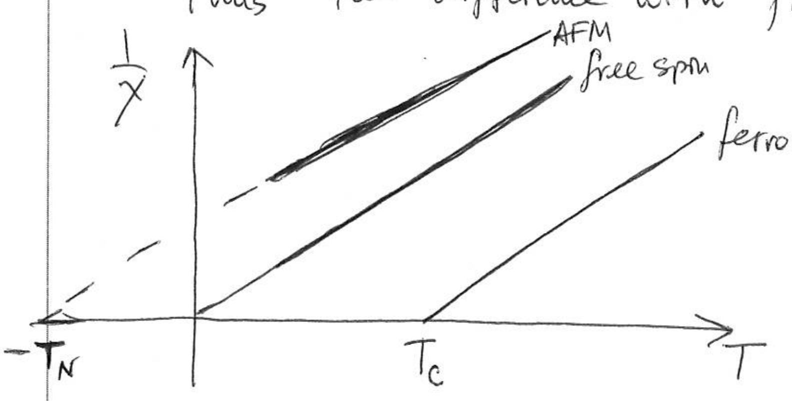
$$\chi_F = \frac{M}{H} = \frac{\mu_0 M}{B} = \frac{n\mu_0\mu_{eff}^2}{3k_B(T - T_c)} = \frac{C}{T - T_c}$$

high-T susceptibility

Curie-Weiss law

where $C = \frac{n\mu_0\mu_{eff}^2}{3k_B}$, $\mu_{eff} = g\mu_B\, S(S+1)$

Thus the difference with free spins where $\chi = \frac{C}{T}$



Antiferromagnet can be obtained by $J \to -J$ change:

$$\chi_{AF} = \frac{C}{T + T_N}$$