Physics 2210
Fall 2015

smartPhysics
16 Rotational Dynamics
17 Rotational Statics
11/16/2015
**Example 16.1 (3/4)**

A round “wheel” of mass $M$, radius $R$ and $I_{CM} = \gamma MR^2$ $(0 < \gamma < 1)$ rolls without slipping down an incline at angle $\phi$ from the horizontal. Find the linear acceleration of the wheel down the incline.

Alternate Solution:

Let $+x$ be down slope and $+y$ point perpendicularly out of the incline. We will chose CCW as the positive rotation sense. For rolling without slipping then we have:

$$v_x = -R\omega, \quad a_x = -R\alpha$$

Where $v_x$ and $a_x$ are the velocity and acceleration components down slope.

Applying Newton’s Second Law on the CM, first in the $y$-direction

$$Ma_y = N - Mg \cos \phi = 0 \quad \rightarrow \quad N = Mg \cos \phi$$

Which is not really used in this problem except possibly to verify that $f_s < \mu_s N$

In the $x$-direction:

$$Ma_x = Mg \sin \phi - f_s \quad \ldots (1)$$

Next we look at rotation about the CM:

$$I_{CM} \alpha = Rf_s \sin(-90^\circ) = -Rf_s$$

*** We have used the fact that, for the purpose of calculating torque, the force of gravity acts at the center-of-mass (CM). This follows from $I_{CM} \alpha = Rf_s \sin(-90^\circ) = -Rf_s$
Example 16.1 (4/4)

A round “wheel” of mass $M$, radius $R$ and $I_{CM} = \gamma MR^2$ $(0 < \gamma < 1)$ rolls without slipping down an incline at angle $\phi$ from the horizontal. Find the linear acceleration of the wheel down the incline.

Solution (continued): From last page

$$Ma_x = Mg \sin \phi - f_s \quad \cdots (1)$$

$$I_{CM} \alpha = Rf_s \sin(-90^\circ) = -Rf_s$$

But $a_x = -R\alpha$, $\alpha = -a_x/R$, and so we have

$$\gamma MR^2 \cdot \frac{-a_x}{R} = Rf_s, \quad \rightarrow \quad -\gamma Ma_x = f_s \quad \cdots (2)$$

Substituting (2) back into (1) for $f_s$:

$$Ma_x = Mg \sin \phi - \gamma Ma_x, \quad \rightarrow \quad (1 + \gamma)Ma_x = Mg \sin \phi$$

And so we have:

$$a_x = \frac{g \sin \phi}{1 + \gamma}$$

Example given in Main Point: Solid Ball/Sphere

$$I = \frac{2}{5} MR^2 \quad \rightarrow \quad \gamma = \frac{2}{5} \quad \rightarrow \quad 1 + \gamma = \frac{7}{5} \quad \rightarrow \quad \frac{1}{1 + \gamma} = \frac{7}{5}$$

$$a_{CM} = \frac{5}{7} g \sin \theta$$

$$a_x = \frac{7}{5} g \sin \phi$$
Example 16.2 (1/6)

A 1152 kg car is being unloaded by a winch. At the moment shown below, the gearbox shaft of the winch breaks, and the car falls from rest. During the car's fall, there is no slipping between the (massless) rope, the pulley, and the winch drum. The moment of inertia of the winch drum is 344 kg·m² and that of the pulley is 3 kg·m². The radius of the winch drum is 0.80 m and that of the pulley is 0.30 m. Find the speed of the car as it hits the water.

Solution: Looking for the speed of something as it drops through a certain distance. Does it remind you of a problem to solve using conservation of energy? Except in this case, as the car acquires speed, both the pulley and the drum acquire angular speed (no slipping!!!):

Drum: \( R_D \omega_D = v_C \) ... (1), Pulley: \( R_P \omega_P = v_C \) ... (2).

The total kinetic energy of the system is the sum of those of the three objects:

\[
K = K_D + K_P + K_C = \frac{1}{2} I_D \omega_D^2 + \frac{1}{2} I_P \omega_P^2 + \frac{1}{2} M_C v_C^2 \quad \ldots (3)
\]

Substituting (1) and (2) into (3) gives us

\[
K = \frac{1}{2} I_D \left( \frac{v_C}{R_D} \right)^2 + \frac{1}{2} I_P \left( \frac{v_C}{R_P} \right)^2 + \frac{1}{2} M_C v_C^2 = \frac{1}{2} \left( \frac{I_D}{R_D^2} + \frac{I_P}{R_P^2} + M_C \right) v_C^2
\]
Example 16.2 (2/6)

A 1152 kg car falls: no slipping between the (massless) rope, the pulley (has mass and rotates with the string), and the winch drum. Moment of inertia of the winch drum: 344 kg·m², of pulley: 3 kg·m². Radius of the winch drum is 0.80 m, pulley: 0.30 m.

Find the speed of the car as it hits the water.

\[ K = \frac{1}{2} \left( \frac{I_D}{R_D^2} + \frac{I_P}{R_P^2} + M_C \right) v_C^2 \]

No friction: total energy is conserved. The only potential energy is the gravitational potential energy of the car: so we have \( \Delta E = \Delta K + \Delta U = 0 \). We started from rest: \( K_i = 0 \)

So the final kinetic energy (after 5.0m drop) is then

\[ K_f = K_i + \Delta K = -\Delta U = -(M_C g \Delta h), \quad \Delta h = -5.0 \text{m} \]

And so

\[ v_{Cf}^2 = \frac{-2M_C g \Delta h}{\frac{I_D}{R_D^2} + \frac{I_P}{R_P^2} + M_C} = \frac{-2 \cdot 1152 \text{kg} \cdot 9.8 \text{ m/s}^2 \cdot (-5.0 \text{m})}{\frac{344 \text{ kg} \cdot \text{m}^2}{(0.80 \text{m})^2} + \frac{3 \text{ kg} \cdot \text{m}^2}{(0.30 \text{m})^2} + 1152 \text{kg}} = \frac{112896 \text{ kg} \cdot \text{m}^2/\text{s}^2}{1723.8 \text{ kg}} = 65.53 \frac{\text{m}^2}{\text{s}^2} \]

\[ v_{Cf} = 8.10 \text{ m/s} \]
Remember from last class, we said that the gravitational potential energy $U_g$ of an object is given by the height of the Center-of-mass. If we take $U_g = 0$ at $y = 0$ (where the $+y$ direction is up), then

$$U_g = MgY_{CM}$$

The main point above is equivalent to this statement about $U_g$.
Torque of Gravitational Force

Let the pivot for the torque calculation be at origin (this is not required).

We think of an extended object as composed of point masses $m_1, m_2, \cdots, m_N$ located at $\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_N$.

Then the torque on the body is the sum of the torque on the point masses (remember this is again a vector equation: it is the shorthand for 3 equations)

$$\vec{\tau} = \sum_{i=1}^{N} \vec{\tau}_i = \sum_{i=1}^{N} (\vec{r}_i \times \vec{F}_i)$$

For gravity, then $\vec{F}_i = m_i \vec{g}$, where $\vec{g}$ is the gravitational acceleration vector (usually written as $\vec{g} = -g \hat{j}$, if we take $+y$ to be up)

$$\vec{\tau} = \sum_{i=1}^{N} (\vec{r}_i \times m_i \vec{g}) = \sum_{i=1}^{N} (m_i \vec{r}_i \times \vec{g}) = \left( \sum_{i=1}^{N} m_i \vec{r}_i \right) \times \vec{g}$$

$$= \left( \frac{1}{M} \sum_{i=1}^{N} m_i \vec{r}_i \right) \times M \vec{g} = \vec{R}_{CM} \times \vec{F}_g$$

Where we have used the fact that cross products are distributive
In case 1, one end of a horizontal massless rod of length $L$ is attached to a vertical wall by a hinge, and the other end holds a ball of mass $M$. In case 2 the massless rod is twice as long and makes an angle of $30^\circ$ with the wall as shown.

In which case is the total torque about an axis through the hinge biggest?

A. Case 1  
B. Case 2  
C. Same
These are 2D problems for Force (x and y-): \[ \sum F_{ix} = 0, \quad \sum F_{iy} = 0, \] and 1D problems for torque (CCW+, CW−): \[ \sum \tau_i = 0 \]
And gives you a total of THREE (3) equations.
An object is made by hanging a ball of mass $M$ from one end of a plank having the same mass and length $L$. The object is then pivoted at a point a distance $L/4$ from the end of the plank supporting the ball, as shown below.

Is the object balanced?

A. Yes  
B. No, it will fall to the left.  
C. No, it will fall to the right.
In case 1, one end of a horizontal plank of mass $M$ and length $L$ is attached to a wall by a hinge and the other end is held up by a wire attached to the wall. In case 2 the plank is half the length but has the same mass as in case 1, and the wire makes the same angle with the plank.

In which case is the tension in the wire biggest?

A. Case 1  
B. Case 2  
C. Same
Unit 17

**MAIN POINTS**

**Lever Arm Calculation of Torque**

The magnitude of any torque can be calculated as the product of the force and its lever arm, the perpendicular distance of a line through this force and the rotation axis.

\[ \tau = r \perp F \equiv (r \sin \theta)F \]

This is exactly the same as \( \vec{\tau} = \vec{r} \times \vec{F} \)

Some people have found the “lever arm” confusing

**A different way of looking at it:**

The **line-of-action** of the force is a line parallel to the direction of the force and passing through the point of application of the force.

The “**Lever arm**”, \( r \perp \), is the perpendicular distance (of closest approach) from the rotation axis to the line-of-action.
Example 17-1

A traffic light hangs from a beam as shown in the figure. The uniform aluminum beam AB is 7.20 m long and has a mass of 12.0 kg. The mass of the traffic light is 21.5 kg. Determine (a) the tension in the horizontal massless cable CD, and (b) the vertical and horizontal components of the force exerted by the pivot A on the aluminum beam.

Solution

This problem is all about balancing all forces and torques on the beam AB. We solve such problems by making a catalogue of all forces acting on the beam, and the torque each exerts.

But first we must decide on a rotation axis about which to calculate torque.

- It is usually convenient to pick a “pivot” that exerts force in both x and y directions on the element in question: because then these forces exert NO torque!!! We pick A

1. Force of pivot: \( F_{px} \) in the x-direction, \( F_{py} \) in the y direction. Notice we are treating them like two separate forces for convenience. They exert no torque (they act at the chosen rotation axis).

2. Tension Force in the cable: \( T \) directed in the \(-x\) direction, The line-of-action is horizontal, The lever arm, \( r_\perp \), is given in the diagram to be 3.80m. The torque (it acts CCW so it is POSITIVE) is \( \tau_T = +r_\perp T = + (3.80 \text{ m}) T \)
Example 17-1

Uniform beam AB: $\ell = 7.20 \text{ m}$, $m = 12.0 \text{ kg}$.  
Traffic light $M = 21.5 \text{ kg}$. 
Determine (a) tension $T$ (b) components $F_{px}$, $F_{py}$ of the force exerted by the pivot on the beam

(3) Weight of the beam itself. $mg$ in the $-y$ direction

Torque: $mg$ acts at radius $r = \ell/2 = 3.60 \text{ m}$. 
Angle from radial vector to weight is $-127^\circ$

So the torque is $\tau_m = (\ell/2)mg \sin(-127^\circ)$: Note the negative orientation of this torque (it wants to deflect beam CW around the pivot) is contained in $\sin(-127^\circ) = -\sin 127^\circ$

(4) Weight of traffic light: $Mg$ in the $-y$ direction

Acts at distance $r = \ell = 7.20 \text{ m}$, and angle from radial vector to weight is AGAIN $-127^\circ$

$\tau_M = \ell mg \sin(-127^\circ)$

Force components in the $x$- and $y$-direction independently sum to zero:

$F_x = F_{px} - T = 0 \quad \ldots \ (1)$

$F_y = F_{py} - mg - Mg = 0 \quad \ldots \ (2)$

(*** note you **CANNOT** add the magnitudes of the forces)

Torques add to zero

$\tau = \tau_T + \tau_m + \tau_M =$

$(3.80 \text{ m})T - (3.60 \text{ m})mg \sin 127^\circ - (7.20 \text{ m})Mg \sin 127^\circ = 0 \quad \ldots \ (3)$
Uniform beam AB: $\ell = 7.20 \text{ m}$, $m = 12.0 \text{ kg.}$

Traffic light $M = 21.5 \text{ kg.}$

Determine (a) tension $T$ (b) components $F_{px}, F_{py}$ of the force exerted by the pivot on the beam.

From equation (3) we have

$$ T = \frac{(3.60 \text{ m})mg \sin 127^\circ + (7.20 \text{ m})Mg \sin 127^\circ}{3.80 \text{ m}} = g \sin 127^\circ \frac{(3.60 \text{ m})m + (7.20 \text{ m})M}{3.80 \text{ m}}$$

$$ = (9.8 \text{ m/s}^2)(0.799) \frac{(3.60 \text{ m})(12.0 \text{ kg}) + (7.20 \text{ m})(21.5 \text{ kg})}{3.80 \text{ m}}$$

$$ = (7.827 \text{ m/s}^2) \frac{198 \text{ kg} \cdot \text{m}}{3.80 \text{ m}} = 408 \text{ N}$$

From equation (1):

$$ F_{px} - T = 0 \quad \rightarrow \quad F_{px} = T = 408 \text{ N}$$

From equation (2):

$$ F_{py} - mg - Mg = 0$$

$$ F_{py} = mg + Mg = (12.0 \text{ kg} + 21.5 \text{ kg})(9.8 \text{ m/s}^2) = 328 \text{ N}$$
Unit 18

Main Points

Gravitational Potential Energy
The gravitational potential energy of an extended object is equal to the product of the weight of the object and the vertical displacement of its center of mass from the height chosen to be the zero of potential energy.

\[ U_{\text{gravity}} = M g Y_{\text{CM}} \]

We already covered this in Unit 16

Condition for Static Equilibrium
The position for static equilibrium of a suspended object can be determined in two equivalent ways:

1) The torque about the suspension axis is zero \( (\tau_{\text{Net}} = 0) \).

2) The gravitational potential energy is minimized \( (dU_{\text{gravity}}/dy = 0) \).

This statement is somewhat misleading: because \( dU_g/dY_{CM} = Mg \) is never ZERO

JUST IGNORE IT!!!!

What we really want to say is

\[ dU_g/d\varphi = 0 \]

To minimize the potential energy

\[ U_g = M g Y_{CM} = M g \frac{\ell}{2} \sin \varphi + b \]
Condition for Stability

The condition for the stability of an extended object placed on a surface is that its center of mass must be located over its footprint on the surface.
Example 17-1 synopsis

A traffic light hangs from a beam as shown in the figure. The uniform aluminum beam AB is 7.20 m long and has a mass of 12.0 kg. The mass of the traffic light is 21.5 kg. Determine (a) the tension in the horizontal massless cable CD, and (b) the vertical and horizontal components of the force exerted by the pivot A on the aluminum beam.

Choose pivot at A to be rotation axis.

1. Force components \( F_{px} \) in the +x direction, \( F_{py} \) in the +y direction. They exert no torque.
2. Tension Force: \( T \) directed in the –x direction. \( \tau_T = +T \cdot (3.80\,\text{m}) \)
3. Weight of the beam: \( mg \) in the –y direction, torque is \( \tau_m = -(\ell/2)mg\sin(127^\circ) \):
4. Weight of traffic light: \( Mg \) in the –y direction, torque is \( \tau_M = \ell Mg\sin(127^\circ) \):

\[
F_x = F_{px} - T = 0 \quad \text{... (1)}
\]
\[
F_y = F_{py} - mg - Mg = 0 \quad \text{... (2)}
\]
\[
\tau = \tau_T + \tau_m + \tau_M = (3.80\,\text{m})T - (3.60\,\text{m})mg\sin 127^\circ - (7.20\,\text{m})Mg\sin 127^\circ = 0 \quad \text{... (3)}
\]

\[
T = (9.8\,\text{m/s}^2)(0.799) \frac{(3.60\,\text{m})(12.0\,\text{kg}) + (7.20\,\text{m})(21.5\,\text{kg})}{3.80\,\text{m}} = 408\,\text{N}
\]

\[
F_{px} - T = 0 \quad \rightarrow \quad F_{px} = T = 408\,\text{N}
\]

\[
F_{py} - mg - Mg = 0
\]
\[
F_{py} = mg + Mg = (12.0\,\text{kg} + 21.5\,\text{kg})(9.8\,\text{m/s}^2) = 328\,\text{N}
\]
Example 18-1

A uniform box of mass \( M = 90 \text{ kg} \), height \( H = 1.2 \text{ m} \) and with \( W = 0.80 \text{ m} \) sits on a rough, horizontal floor. You can apply a horizontal force \( F_A \) at a single point on the left side of the box (say, at distance \( h \) above the floor). The static coefficient of friction between the box and the floor is \( \mu_s = 1.10 \). (a) Find the minimum force \( F_A \) needed to move the box. (b) Find the maximum distance \( d \) above the floor at which you can apply this minimum force without tipping the box over.

We start with a free-body diagram of the box. The additional forces are

1. The weight of the box \( Mg \)
2. The normal force \( N \)
3. The static friction force

We start by balancing the forces, assuming the system is just on the verge of slipping, so that we still have static equilibrium:

\[
F_y = N - Mg = 0, \quad \rightarrow \quad N = Mg
\]
\[
F_x = F_A - f_s = 0, \quad \rightarrow \quad F_A = f_s
\]

We are on the verge of slipping \( \Rightarrow f_s = \mu_s N = \mu_s Mg \), and

\[
F_A = \mu_s Mg = (1.10)(90 \text{ kg})(9.8 \text{ m/s}^2) = 970 \text{ N}
\]
Example 18-1

\( M = 90 \text{ kg}, \ H = 1.2 \text{ m}, \ W = 0.80 \text{ m}, \ \mu_s = 1.10. \)

(a) Find the minimum force \( F_A \) needed to move the box.

(b) Find the maximum \( h \) above the floor at which you can apply this minimum force without tipping the box over.

We now examine the torques exerted by the forces about the LOWER RIGHT CORNER (it is the pivot for tipping).

(1) Applied force: \( \tau_A = -hF_A \)

We have used a trick here: in finding the torque \( \tau = rF \sin \varphi \), we can also try to identify the distance \( r \sin \varphi \), which is often easy to quantify.

Here we have \( r \sin(180^\circ - \varphi) = r \sin \varphi = h \)

(2) Weight: \( \tau_M = +(W/2)Mg \)

Here we have \( r \sin(180^\circ - \varphi) = r \sin \varphi = W/2 \)

**Where** do the normal force and the friction forces act?

The effective point of action lies somewhere on the bottom surface of the box. We think of this as the “weight” of box “shifting” as we increase the force \( F_A \)
Example 18-1

\(M=90 \text{ kg}, \ H=1.2 \text{ m}, \ W=0.80 \text{ m}, \ \mu_s=1.10.\)

(a) Find the minimum force \(F_A\) needed to move the box.

(b) Find the maximum \(h\) above the floor at which you can apply this minimum force without tipping the box over.

If \(F_A = 0\) then the normal forces (and hence the friction) acts at the center of the bottom: the torques from the weight and normal force cancel exactly. As we increase \(F_A\), the point of action of the normal force (and hence the friction) shifts to the right.

When on the verge of tipping, they act at the lower right corner: hence they exert no torque for part (b)

Setting the net torque about lower-right corner to zero:

\[
\tau = \tau_A + \tau_M = -hF_A + (W/2)Mg = 0
\]

\[
h = \frac{(W/2)Mg}{F_A} = \frac{(0.40 \text{ m})(90 \text{ kg})(9.8 \text{ m/s}^2)}{970 \text{ N}}
\]

\[= 0.364 \text{ m}\]

Answer (a) minimum force of 970 N

(b) Maximum height (with 970 N) of 0.364 m