1. Two point charges, \(-q\) and \(\frac{4}{9}q\), are separated by distance \(a\), as shown in the figure at the right.

(a) In terms of \(a\), specify clearly the location of a point where the electric field due to these two charges is zero. \([11]\)

(b) A third charge, \(Q\), is now placed at the neutral point found in part (a). In terms of \(q\), what must be the sign and magnitude of \(Q\) so that the net force on the charge \(-q\) is zero? (In fact, all three charges will be in equilibrium.) Is this a case of stable or unstable equilibrium? \([11]\)

(a) The neutral point must be some distance \(d\) to the right of the charge \(\frac{4}{9}q\), since that is the weaker of the two charges. Equating the magnitudes of the two electric fields at that point, we have

\[
\frac{k_e \frac{q}{(a + d)^2}}{d^2} = \frac{k_e \left(\frac{4}{9}q\right)}{d^2} \Rightarrow \frac{1}{(a + d)^2} = \frac{4}{9d^2} \Rightarrow 9d^2 = 4(a + d)^2.
\]

Taking square roots of each side, and knowing that \(d\) must be positive, we have

\[
3d = 2(a + d), \quad \text{or} \quad 3d = 2a + 2d \Rightarrow d = 2a.
\]

(b) Equating the magnitudes of the two forces acting on \(-q\):

\[
\frac{k_e q}{(a + d)^2} = \frac{k_e \left(\frac{4}{9}q\right)}{a^2}, \quad \text{or} \quad \frac{Q}{(a + 2a)^2} = \frac{4q}{9a^2} \Rightarrow \frac{Q}{(3a)^2} = \frac{4q}{9a^2},
\]

so as far as magnitude is concerned, \(Q = 4q\). Taking signs into account, \(Q = -4q\), since \(Q\) must repel \(-q\) to counteract the attraction of \(\frac{4}{9}q\).

The equilibrium is unstable, broken if any of the three charges is displaced either left or right.
2. (I) In the air over a particular region at an altitude of 150 m above the ground the electric field is 180 N/C directed vertically upwards. At 270 m above the ground the electric field is 140 N/C upwards. What is the average volume charge density in the layer between these two elevations? Is it positive or negative? (Recall that $\varepsilon_0 \approx 8.85 \times 10^{-12}$ C$^2$/N·m$^2$.) [8]

(II) A long, solid, non-conducting circular cylinder of radius R has volume charge density $\rho$ that varies with radial distance $r$ from the axis: $\rho = B/r$, for $r \leq R$, where B is a positive constant.

(a) What are the SI units of B? [2]

(b) Given your answers in terms of $\varepsilon_0$ [instead of $k_e$, which is $1/(4\pi\varepsilon_0)$], use Gauss’s law to find the magnitude of the electric field $E$ for: (i) $r < R$ [10]; (ii) $r > R$. [4]

\[
\text{Gauss's Law: } \oint_{S} \mathbf{E} \cdot d\mathbf{A} = \frac{q_{\text{enc}}}{\varepsilon_0} \Rightarrow \frac{q_{\text{enc}}}{\varepsilon_0} = \frac{-40 \pi}{\varepsilon_0} \Rightarrow \rho = \frac{q_{\text{enc}}}{\varepsilon_0} = -1 \frac{40 \pi}{120 \text{ m}} \Rightarrow \rho = -2.95 \times 10^{-12} \frac{\text{C}}{\text{m}^3}
\]

(b) (i) Consider a cylindrical shell of radius $r$, length $l$, and thickness $dr$. Its volume is $dV = 2\pi rl \, dr$, and the charge it contains is $dq = \rho \, dV = \frac{B}{r} (2\pi rl \, dr) = 2\pi Bl \, dr$. The total charge contained inside a Gaussian cylinder of length $l$ and radius $r < R$ is therefore $q_{\text{in}} = \int_0^r 2\pi Bl \, dr = 2\pi Blr$. Gauss's law then gives

\[
E(2\pi rl) = \frac{2\pi Blr}{\varepsilon_0} \Rightarrow E = \frac{B}{\varepsilon_0} \text{ (The radial electric field has constant magnitude)}.
\]
(ii) Now \( q_{in} = \int_0^R 2\pi Br \, dr = 2\pi Br R \), Gauss's law now gives

\[
E(2\pi r R) = \frac{2\pi Br R}{\varepsilon_0} \Rightarrow E = \frac{BR}{\varepsilon_0 r} \quad \text{for} \quad r > R
\]

Notice that at \( r = R \) the expressions in (i) and (ii) agree.
3. Three point charges are fixed in place on the y-axis, as shown in the sketch at the right. A charge +6q is at the origin, −q is at the point with coordinates (0, 3a), and +4q is at (0, −3a).

(a) Giving your answer in terms of ke, a, and q, how much work was done to assemble these charges? In other words, what is the total electrostatic potential energy U of the configuration? [8]

(b) What is the electrostatic potential V at the point P whose coordinates are (4a, 0)? [8]

(c) A particle of mass m and charge +2q is now moved in from infinity and held in place at point P.

(i) How much extra potential energy has this added to the system? [3]

(ii) If the particle at point P is released from rest, find its speed, v (in terms of ke, a, m, and q), after it has moved freely to a very large distance away. [7]

\[(a) \quad U = \frac{ke^2 q_1 q_2}{r}, \text{ so the total electrostatic potential energy of the array of three charges is } U = ke \left[ \frac{(-q)(6q)}{3a} + \frac{(q)(4q)}{6a} + \frac{(4q)(q)}{3a} \right],\]

or \[U = ke \left( \frac{-6q^2}{3a} - \frac{2q^2}{3a} + \frac{24q^2}{3a} \right) = \frac{16q^2 ke}{3a}\]

\[(b) \quad V = \frac{keq}{r}, \text{ so at P, } V = -\frac{keq}{5a} + \frac{6qke}{4a} + \frac{4qke}{5a}, \text{ or } V = ke \left[ \frac{-2q}{10a} + \frac{15q}{10a} + \frac{8q}{10a} \right] \Rightarrow V = \frac{21keq}{10a}.\]

(c) (i) Placing the charge 2q at P adds an extra potential energy (2q)·V = (2q)·\(\frac{21keq}{10a}\) = \(\frac{21keq^2}{5a}\) to the system.

(ii) The extra potential energy added in (i) now becomes kinetic energy, so \[\frac{1}{2}mv^2 = \frac{21keq^2}{5a} \Rightarrow v^2 = \frac{42keq^2}{5ma} \Rightarrow v = \sqrt{\frac{42keq^2}{5ma}}.\]
Physics 2220, Summer 2017

First Exam

4. Three capacitors (1.0 \( \mu F \), 2.0 \( \mu F \), and 1.0 \( \mu F \)), a 12-V battery, and a switch, S, are connected as shown in the sketch at the right. (In the following, for any results that do not come out as whole numbers, it is preferable to express the answers as fractions in lowest terms instead of as decimals.)

(a) Calculate the equivalent capacitance of the array of capacitors between points \( a \) and \( b \). [4]

(b) The switch S is closed for a while and then reopened. Find the charge, the potential difference (p.d.), and the electrostatic potential energy \( U \) for each of the individual capacitors, filling in the table provided below with the answers. [9]

<table>
<thead>
<tr>
<th>capacitor</th>
<th>charge (in ( \mu C ))</th>
<th>p.d. (in V)</th>
<th>energy (in ( \mu J ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>upper 1.0 ( \mu F )</td>
<td>8.0</td>
<td>8.0</td>
<td>32</td>
</tr>
<tr>
<td>2.0 ( \mu F )</td>
<td>8.0</td>
<td>4.0</td>
<td>16</td>
</tr>
<tr>
<td>lower 1.0 ( \mu F )</td>
<td>12</td>
<td>12</td>
<td>72</td>
</tr>
</tbody>
</table>

(c) With S still open, an external agent inserts a slab of material of dielectric constant \( \kappa = 6.0 \) into the upper 1.0-\( \mu F \) capacitor, so that the space between its plates is filled. Now recalculate the quantities asked for in (b). [Hint: With the dielectric inserted, the potential difference between points \( a \) and \( b \) will no longer be 12 V as it was in part (b).] [12]

<table>
<thead>
<tr>
<th>capacitor</th>
<th>charge (in ( \mu C ))</th>
<th>p.d. (in V)</th>
<th>energy (in ( \mu J ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>upper 1.0 ( \mu F ) (now with dielectric)</td>
<td>12</td>
<td>2.0</td>
<td>12</td>
</tr>
<tr>
<td>2.0 ( \mu F )</td>
<td>12</td>
<td>6.0</td>
<td>36</td>
</tr>
<tr>
<td>lower 1.0 ( \mu F )</td>
<td>8.0</td>
<td>8.0</td>
<td>32</td>
</tr>
</tbody>
</table>

(d) Compare the total energies stored in the capacitors in parts (b) and (c), and account physically for any difference. [3]

\[ \begin{align*}
\text{In (b)}: \quad & 32 + 16 + 72 = 120 \mu J \\
\text{In (c)}: \quad & 12 + 36 + 32 = 80 \mu J \\
\text{difference} & = 40 \mu J
\end{align*} \]

The agent that inserted the dielectric did -40 \( \mu J \) of work. The charges on the capacitor attracted the dielectric, doing 40 \( \mu J \) of work, so the system lost 40 \( \mu J \) of electrostatic potential energy.
4. (cont’d)

(a) In series: \[ \frac{1}{C_{eq}} = \frac{1}{1} + \frac{1}{2} = \frac{3}{2} \Rightarrow C_{eq} = \frac{2}{3} \text{ mF}. \] This is in parallel with the 1.0 \text{ mF} capacitor, so the total equivalent capacitance is \[ \frac{2}{3} + 1 = \boxed{\frac{5}{3} \text{ mF}}. \]

(b) The total amount of charge separated by the battery is (\(C_{eq}\))\(V\) = (\(\frac{5}{3} \text{ mF}\))(12V) = \(20\mu\text{C}\). The charge on the lower 1.0 \text{ mF} capacitor is (1\mu\text{F})(12V) = 12\mu\text{C}, so there must be \(20 - 12 = 8\mu\text{C}\) on each of the upper capacitors (in series). Use \(V = \frac{Q}{C}\) to find the p.d. across each of the upper capacitors, and use \(\frac{Q^2}{2C}\) (or \(\frac{1}{2}CV^2\)) to find the energy in each capacitor.

(c) The upper capacitor that was formerly 1.0 \text{ mF} is now 6.0 \text{ mF}, so \[ \frac{1}{C_{eq}} = \frac{1}{6} + \frac{1}{2} = \frac{1}{6} + \frac{3}{6} = \frac{4}{6} = \frac{2}{3} \Rightarrow C_{eq} = \frac{3}{2} \text{ mF} \] for the series combination, so the total equivalent capacitance is now \(\frac{3}{2} + 1 = \boxed{\frac{5}{2} \text{ mF}}\). Since the battery has been disconnected (S is open), the total charge must still be the same as before, 20\mu\text{C}, but the p.d. from a to b is now less than before: \(V = \frac{Q}{C} = \frac{20\mu\text{C}}{\frac{5}{2} \text{ mF}} = 8.0\text{V}\), which is also the p.d. across the lower 1.0 \text{ mF} capacitor, which means that its charge is \(Q = CV = (1\mu\text{F})(8\text{V}) = 8.0\mu\text{C}\). Each of the upper (series) capacitors must then have a charge of \(20 - 8 = 12\mu\text{C}\). For the 6.0 \text{ mF} capacitor, \(V = \frac{Q}{C} = \frac{12\mu\text{C}}{6 \text{ mF}} = 2.0\text{V}\), and for the 2.0 \text{ mF} capacitor, \(V = \frac{12\mu\text{C}}{2 \text{ mF}} = 6.0\text{V}\). The energy in each capacitor is again found from either \(\frac{Q^2}{2C}\) or \(\frac{1}{2}CV^2\).