Physics 2220  
First Exam  
Name: **SOLUTIONS**

Summer 2018  
(Chs 23–26)  
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Friday 22 June  
Circle your Discussion TA:  
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You may use your one sheet of notes and formulas, but do not collaborate with any other person. Do all four problems, showing your method and working clearly (a right answer alone is not necessarily enough). Include correct SI units in your answers where appropriate. The number of marks for each portion of a question is given in square brackets, [ ], to its right.

I. (a) Two very small, identical conducting spheres are held in place at distance \( r \) apart. One sphere has electric charge \( 14q \), and the other has charge \( -2q \).

(i) In terms of \( q \), \( r \), and the Coulomb constant \( k_c \), find the magnitude of the electrostatic force between the two spheres. Is it attractive or repulsive? [4]

(ii) Now the spheres are connected momentarily by a thin conducting wire. Answer the same questions as in part (i) after equilibrium is re-established. [6]

(b) A thin insulating rod of length \( l \) carrying uniform positive linear charge density \( \lambda \) lies along the positive \( y \)-axis with its lower end at the origin, as in the sketch at the right. \( P \) is the point on the \( x \)-axis whose coordinates are \((x, 0)\). Use integration to find the \( y \)-component of the electric field vector \( \mathbf{E} \) at point \( P \). [14]

\[
\begin{align*}
\text{(a) (i)} & \quad F = \frac{k_c (14q)(2q)}{r^2} = 28 \frac{k_c q^2}{r^2}, \quad \text{(attractive)} \\
\text{(ii) The net charge is } 14q - 2q = 12q, \text{ which is shared equally} & \quad \text{between the two identical spheres, so } F = \frac{k_c (6q)(6q)}{r^2} = \frac{36 k_c q^2}{r^2}, \quad \text{(repulsive)} \\
\text{(b) The typical charge element } \text{d}q = \lambda \text{d}y \text{ shown in the sketch produces} & \quad \text{electric field } \text{d}E = \frac{k_c \lambda \text{d}y}{r^2} = \frac{k_c \lambda \text{d}y}{\sqrt{x^2 + y^2}} \text{ at point } P, \text{ and its } y \text{-component} \\
\text{is } \text{d}E_y = -\text{d}E (\cos \theta) = -\text{d}E \left( \frac{y}{r} \right) & \quad \text{, or } \text{d}E_y = -\frac{k_c \lambda \text{d}y}{(x^2 + y^2)^{3/2}}. \quad \text{Integrating:} \\
E_y = \int \text{d}E_y = -\frac{k_c \lambda}{\sqrt{x^2 + y^2}} \int_0^l \frac{y \text{d}y}{(x^2 + y^2)^{3/2}} \quad \text{. Letting } u = x^2 + y^2 \Rightarrow \text{d}u = 2y \text{d}y, \text{ this} \\
\text{becomes } E_y = -\frac{k_c \lambda}{2} \int_{x^2}^{x^2 + l^2} u^{-3/2} \text{d}u & = -\frac{k_c \lambda}{2} \left[ \frac{1}{(-\frac{1}{2})} \right] \left| \frac{1}{x^2} \right| = k_c \lambda \left[ \frac{1}{x \sqrt{x^2 + y^2}} - \frac{1}{l \sqrt{x^2 + y^2}} \right], \quad \text{or } E_y = -k_c \lambda \left( \frac{1}{x \sqrt{x^2 + y^2}} - \frac{1}{l \sqrt{x^2 + y^2}} \right) \\& \text{or } E_y = k_c \lambda \left[ \frac{x}{x \sqrt{x^2 + y^2}} - \frac{1}{x} \right] \quad \text{, or } \quad E_y = -\frac{k_c \lambda}{x} \left( 1 - \frac{x}{\sqrt{x^2 + y^2}} \right) \quad \text{.}
\end{align*}
\]
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2. (I) In the figure shown at the right, an empty butterfly net is totally immersed in a uniform electric field \( \mathbf{E} \) of magnitude \( E \). The opening of the net, a circle of radius \( a \), is perpendicular to \( \mathbf{E} \). Use Gauss’s law to find the flux of \( \mathbf{E} \) through the net only (not through its circular opening). [8]

(II) An insulating solid ball of radius \( R \) has non-uniform but spherically symmetric positive charge density \( \rho = \frac{A}{r} \) throughout its interior volume \((0 \leq r \leq R)\), where \( r \) is the radial distance from the centre of the ball, and \( A \) is a positive constant.

(a) What are the SI units of \( A \)? [2]

(b) In terms of \( A, r, R, \) and \( \varepsilon_0 \), calculate:
   (i) the electric field magnitude, \( E \), for \( r < R \); [11]
   (ii) the total charge \( Q \) contained in the ball; [2]
   (iii) the electric field magnitude, \( E \), for \( r > R \); [3]

(I) Because the net is empty, containing no electric charge, Gauss’s law requires that the total flux of \( \mathbf{E} \) through its closed surface must be zero. That is, \( \Phi_{\text{netting}} + \Phi_{\text{opening}} = 0 \Rightarrow \Phi_{\text{netting}} = -\Phi_{\text{opening}} \), or \( \Phi_{\text{netting}} = -\pi a^2 E \).

(II) (a) \( \rho = \frac{A}{r} \Rightarrow A = \rho r \). Now \( \rho \) is in \( \text{C} \text{m}^{-3} \) and \( r \) is in \( \text{m} \), so \( A \) must be in \( \text{C} \text{m}^{-2} \text{m} \), or \( \frac{\text{C}}{\text{m}^2} \).

(b) The charge \( dq \) in a thin spherical shell of radius \( s \) and thickness \( ds \) is \( dq = 4\pi s^2 \rho \, ds = \left(\frac{A}{s}\right) (4\pi s^2 \, ds) \), or \( dq = 4\pi As \, ds \). Then the total charge in a ball of radius \( r < R \) is \( q_{\text{in}} = \int_0^r 4\pi As \, ds = 2\pi Ar^2 \), or \( q_{\text{in}} = 2\pi Ar^2 \). From Gauss’s law, \( \oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{in}}}{\varepsilon_0} \),

\[ \mathbf{E} \cdot 4\pi r^2 = \frac{2\pi Ar^2}{\varepsilon_0} \Rightarrow E = \frac{A}{2\varepsilon_0} \] (i.e., an electric field of constant magnitude)

(ii) Putting \( r = R \) in the result above, \( Q = 2\pi AR^2 \).

(iii) For \( r > R \): \( E(4\pi r^2) = \frac{Q}{\varepsilon_0} = \frac{2\pi AR^2}{\varepsilon_0} \), so \( E = \frac{AR^2}{2\varepsilon_0 r^2} \), which agrees with the result in (ii) when \( r = R \).
3. (a) Two point charges are held in place on the $x$-axis, as shown in the sketch at the right. Charge $+10q$ is at the origin, and $+15q$ is at the point with coordinates $(4a, 0)$. $P$ is the point whose coordinates are $(4a, 3a)$.

(i) Giving your answer in terms of $k\varepsilon$, $a$, and $q$, what is the electrostatic potential, $V$, at point $P$? \[ \text{[8]} \]

(ii) If a particle of charge $+2q$ is now moved in from infinity to point $P$, what will be the total electrostatic potential energy, $U$, of the system of three charges? \[ \text{[4]} \]

(b) Consider two hard, solid, insulating balls, each of radius $R$ and mass $M$, held initially at a centre-to-centre separation of $50R$. One of the balls has electric charge $Q$ distributed uniformly throughout its volume, and the other has charge $-2Q$ similarly distributed. The balls are released from rest and move towards each other along the line joining their centres, under the influence of electrostatic attraction alone (that is, ignore any gravitational or frictional forces). At the instant they collide, each ball will be moving at the same speed $v$. Use conservation of energy to find the value of $v$ in terms of $Q$, $R$, $M$, and $k\varepsilon$, simplifying your answer algebraically. \[ \text{[12]} \]
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4. Four capacitors (20.0 \( \mu \)F, 10.0 \( \mu \)F, 40.0 \( \mu \)F, and 12.0 \( \mu \)F), together with an open switch, S, are connected as shown in the sketch at the right. The 20.0-\( \mu \)F capacitor is charged to a potential difference of 12.0 V, whereas the other three capacitors are initially uncharged.

(a) Calculate the charge on the 20.0-\( \mu \)F capacitor, and the electrostatic energy stored in it. [5]

(b) Find the equivalent capacitance of the system of the three uncharged capacitors. [6]

(c) After the switch S is closed, calculate the charge, the potential difference (p.d.), and the electrostatic potential energy \( U \) for each of the individual capacitors, filling in the table provided below with the answers. [12]

<table>
<thead>
<tr>
<th>capacitor</th>
<th>charge (in ( \mu )C)</th>
<th>p.d. (in V)</th>
<th>energy (in ( \mu )J)</th>
</tr>
</thead>
<tbody>
<tr>
<td>20.0 ( \mu )F</td>
<td>120</td>
<td>6</td>
<td>360</td>
</tr>
<tr>
<td>10.0 ( \mu )F</td>
<td>48</td>
<td>4.8</td>
<td>115.2</td>
</tr>
<tr>
<td>40.0 ( \mu )F</td>
<td>48</td>
<td>1.2</td>
<td>28.8</td>
</tr>
<tr>
<td>12.0 ( \mu )F</td>
<td>72</td>
<td>6</td>
<td>216</td>
</tr>
</tbody>
</table>

(d) Compare the total electrostatic potential energies stored in the capacitors in parts (a) and (c), and try to account physically for any difference. [3]

\[ q = CV = (20 \mu F)(12V) = \frac{240}{10} \mu C; \quad U = \frac{1}{2} CV^2 = \frac{1}{2}(20 \mu F)(12V)^2 = 1440 \mu J. \]

(b) 10\( \mu \)F and 40\( \mu \)F in series: \( \frac{1}{C} = \frac{1}{10} + \frac{1}{40} = \frac{4}{40} + \frac{1}{40} = \frac{5}{40} \Rightarrow C = \frac{40}{5} = 8 \mu F. \]

That is, the array of 3 capacitors can be reduced to \( 8 \mu F \) in parallel \( \leftarrow \frac{1}{8 \mu F} = 12 \mu F \leftarrow \frac{1}{10 \mu F} \leftarrow \frac{1}{40 \mu F} \leftarrow \frac{1}{12 \mu F} = \frac{1}{(8 \mu F + 12 \mu F)}. \]

(c) The original charge of 240\( \mu \)C will distribute equally between the original 20\( \mu \)F capacitor and the 3-capacitor array whose equivalent capacitance is also 20\( \mu \)F. The p.d. across the 20\( \mu \)F capacitor is then \( V = \frac{Q}{C} = \frac{120}{20} \mu F = 6.0 V \), as it is also across the 12\( \mu \)F capacitor in parallel, whose charge must be \( (12 \mu F)(6V) = 72 \mu C. \) The remaining 120\( \mu C - 72 \mu C = 48 \mu C \) must be on the 10\( \mu \)F and 40\( \mu \)F capacitors (same charges, since in series). Knowing the charges, we can easily calculate the p.d.s and energies. (d) The total energy after is \( 360 + 115.2 + 28.8 + 216 = \frac{720}{\mu J} \), just half of the original. The 'lost' energy goes to electromagnetic radiation and thermal energy in the connecting wires. See Example 26.5.