To analyze the rocket problem choose coordinate systems as shown.

\[ S \quad  \quad  S' \]

\[ A \quad  \quad  B \]

\[ \quad x \quad  \quad  x' \]

We have:

\[ x_{TA} = -L \quad x_{TB} = 0 \]
\[ x_{UA} = 0  \quad x_{UB} = -L \]

The velocity of \( S' \) relative to \( S \) is \(-\alpha\).

Hence:

\[ x' = \alpha (x + \alpha t) \quad x = \alpha (x' - \alpha t') \]
\[ t' = \alpha (t + \frac{\alpha \eta}{c^2}) \quad t = \alpha (t' - \frac{\alpha \eta}{c^2}) \]

As seen from \( A \):

\[ x_{UB} = \alpha (x_{UB}' - \alpha x_{BO}') \]
\[ t_{UB}' = \alpha (t_{BO} + \frac{\alpha x_{BO}'}{c^2}) \]

\[ x_{UB} = \alpha (x_{UB}' - \alpha \alpha [x_{BO} + \frac{\alpha x_{BO}'}{c^2}]) \]
\[ x_{UB}' (1 + \frac{\alpha^2}{c^2}) = \alpha (x_{UB}' - \alpha \alpha x_{BO}') \]
\[ x_{UB} [1 + \frac{\alpha^2}{c^2}] = \alpha (x_{UB}' - \alpha x_{BO}') \]
\[ x_{0B} = x \left[ -\frac{v_0}{c} - 0 \right] \]

\[ x_{0B} = -\frac{v_0}{c} \]

Hence a miss by \( \ell_0(1 - \frac{1}{c^2}) = \frac{v_0}{c} - \frac{v_0}{c} \)

As seen by \( A \):

\[ x_{TA} = x \left( x_{TA} + c x_{TA} \right) \]

\[ = x \left( -\frac{v_0}{c} + 0 \right) \]

A miss by \( x_{0B} - x_{0A} = (x - 1) \ell_0 \)

Hence both agree it is a miss. Note that the misses obey \( \ell = \frac{v_0}{c} \) as expected.

But the reason for the miss appears different in the two systems. As seen from \( A \) it is because \( B \) has shrunk:

\[
\begin{array}{cc}
1 & B \\
\rightarrow & \\
A & \\
\end{array}
\]

As seen from \( B \) it is because the gunner fired too soon:

\[ x_{\text{AT}} = x \left( x_{\text{AT}} + \frac{v}{c^2} x_{\text{AT}} \right) = x \left( 0 - \frac{v_0}{c^2} \right) + 0 \]
This is characteristic of "paradoxes". They arise when we mistakenly assume two events which are simultaneous in one system are simultaneous in another. In this case the events are nose reads tail and gun fires. Since they are separated by a distance $s$, in $S$ they can't be simultaneous in $S'$ if they are in $S$.

**Velocity Addition**

Consider an object moving with speed $u'$ in system $S'$ as shown.

\[
\begin{array}{c}
S' \\
\hline
x' \\
\hline S
\end{array}
\]

What velocity will $S$ measure. This is easily found from the Lorentz equations:

\[
x = x'(x' + ut')
\]

\[
t = x'(t' + ut')
\]

Take the differential of both sides of each equation:
\[ \frac{dx}{dt} = \beta (\frac{dx'}{dt'} + \frac{u}{c^2} \frac{dx}{dt}) \]

\[ \frac{dt}{dt} = \beta (\frac{dt'}{dt'} + \frac{u}{c^2} \frac{dx}{dt}) \]

Now divide equation 1 by 2:
\[ \frac{dx}{dt} = \frac{dx'}{dt'} + \frac{u}{c^2} \frac{dx}{dt} = \frac{\frac{dx'}{dt'} + \frac{u}{c^2} \frac{dx}{dt}}{1 + \frac{u}{c^2} \frac{dx}{dt}} \]

Now let \( \frac{dt}{dt} \to 1 \). Then:
\[ \frac{dx}{dt'} = u' \]
\[ \frac{dx}{dt} = u \]

and:
\[ u = \frac{u' + u}{1 + \frac{u}{c^2} u} \]

If instead the velocity was in the opposite direction we would have:
\[ dy = dy' \]
\[ dt = \beta (\frac{dt'}{dt'} + \frac{u}{c^2} \frac{dx}{dt}) \]

\[ \frac{dy}{dt} = \frac{dy'}{dt'} = \frac{\frac{dy'}{dt'}}{\beta (1 + \frac{u}{c^2} \frac{dx}{dt})} = \frac{u'}{u} \]

Note that in either case \( u \) is always \( < c \). For example if \( u = 0.9c \) and \( \beta = 0.9c \) we get:
\[ U = \frac{9c + 8c}{1 + 0.81} = 0.994c \]

**Twin Paradox**

We can now consider the most famous paradox of all. Suppose we have 2 twins. One stays on the earth while the other goes on a space trip in which she accelerates away from the earth at \( g \) for 10 years, decelerates at \( g \) for 10 years, and then reverses the process to return home (times as seen by her). For her 40 years will have passed, but because of time dilation (\( \Delta t = 80\% \)) a longer time will have passed for the turn on earth. But as seen by the travelling twin the reverse is true. She sees the earth bound twin as moving and hence aging more slowly. When they reunite after the trip, which will be older?

Back in the middle ages when I was a graduate student this was a controversial topic. Many physicists believed that
Special Relativity only applied to unaccelerated observers because the Lorentz equations applied to those. Here they did not know how to resolve the paradox mathematically. Physically it was easier — the two observers are not equivalent since one is accelerated and one is not.

However, as we have seen, nature never cares who looks. The restriction on Special Relativity has nothing to do with acceleration. The limitation is that space must be flat — no gravity. In most cases gravity is small (unless you happen to be very close to something very massive) and we can safely use SR. The question is how to do it when one system is accelerating.

The trick is to imagine space filled with observers moving at constant (but different speeds). Then at any instant the space twin will have the same velocity as one of those systems. For that instant
we can apply the Lorentz transformation to 2 systems and the earth system. For the next instant the transformation will be between the earth system and another space system - etc.

Suppose the space turn has an acceleration (as measured by der) of \(a(t)\) where \(t'\) is her time. Then consider the velocity as seen by the earth turn.

\[
\begin{align*}
\mathbf{E} & \quad \mathbf{S} \\
\rightarrow & \quad \mathbf{U}(t')
\end{align*}
\]

where \(S\) is the inertial space system which happens to have the velocity of the space turn at \(t'\). Then to find the velocity at \(t'=ct'\) as seen from earth we must simply do the velocity addition discussed above:

\[
u(t'+dtt) = \frac{\nu(t) + adt'}{1 + \nu(t)adt'/c^2}
\]
Then:
\[ u(t + \Delta t)^2 \left( 1 + \frac{u(z)^2}{c^2} \right) = u(z) + \Delta u(z) \]
\[ u(t + \Delta t) - u(t) = \Delta t \left[ 1 - \frac{u(x)^2}{c^2} \right] \]
\[ \frac{u(t + \Delta t) - u(t)}{\Delta t} = \Delta t \left[ 1 - \frac{3u(x)^2}{c^2} \right] \]
Now let \( \Delta t \rightarrow 0 \). Then:
\[ \frac{du(z)}{dt} = \theta \left[ 1 - \frac{u(z)^2}{c^2} \right] \]
\[ \frac{du(z)}{dt} = \theta u(z) \frac{dt}{dt} \]
\[ \frac{d}{d\xi} \left( \frac{u(z)}{1 - \frac{u(z)^2}{c^2}} \right) = \int_0^\xi \theta u(z) \, d\xi \]
\[ \cosh \left( \frac{u(z)}{c} \right) = \int_0^\xi \theta u(z) \, d\xi \rightarrow \alpha t \]
\[ \tanh \left( \frac{u(z)}{c} \right) \rightarrow \frac{\alpha t}{c} \]
\[ \frac{u(z)}{c} = \frac{\alpha t}{c} \]
\[ \frac{d}{dt} = + \text{mark} \left( \frac{\alpha t}{c} \right) \]
\[ x(t') = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \]

\[ = \frac{1}{\sqrt{1 - \tanh^2\left(\frac{v}{c}\right)}} \]

\[ = \frac{1}{\sqrt{\frac{sech^2\left(\frac{v}{c}\right)}{1}}} = \cosh\left(\frac{ct'}{c}\right) \]

But we know:

\[ dt = \gamma dt' \]

\[ \gamma \int_0^t dt' = \int_0^t x(t') \, dt' \]

\[ t = \gamma \int_0^{ct'} \cosh\left(\frac{ct'}{c}\right) \, dt' \]

\[ = \frac{c}{\alpha} \int_0^{ct'} \cosh(\alpha t') \, dt' \]

\[ t = \gamma \frac{c}{\alpha} \sinh\left(\frac{ct'}{c}\right) \]

For ces: \( T = 10^{12} \text{ years} \), \( \alpha = 9.8 \)

Thus:

\[ t_E = 4 \times 10^8 \sinh\left(\frac{9.8 \times 10^{12}}{2 \times 10^8}\right) \]

\[ t_E = 1.76 \times 10^{12} \text{ years} = 5.67 \times 10^{4} \text{ years} \]