Physics 3210: Space-time diagrams.

Before we talk about space-time diagrams, we have to talk about how we establish distance and how we synchronize clocks.

In relativity, the laws of physics must remain the same for all non-accelerating observers. This in turn means that Maxwell's equations for electromagnetics also remain valid—hence, everyone measures the same speed of light in a vacuum. This result was shown in Michelson-Morley's experiment.

So... first, distance.

In a particular system S, we'll place a mirror some distance d away and send a beam of light to that mirror. Since the speed of light is the same going to the mirror and coming back from the mirror, we have:

\[ d = \frac{1}{2} cT \]

where T is the time interval we measured for the round trip.

In a space-time diagram, this same situation would look like this:
Next, we have to figure out how to synchronize clocks.

This procedure will also use the fact that the speed of light is the same for all observers.

Again we place a mirror some distance $d$ away. We also place a clock there.

Now we send a beam of light to the mirror and back. The round trip time was $T$, so we say that at time $\frac{T}{2}$, the light hit the mirror. In a space-time diagram:

Now that we’ve established these things, we can study space-time diagrams. We will use the $k$-calculus method.

Suppose there are 2 observers. One is in the $S$ system, and the other in the $S'$ system is moving with velocity $v$ with respect to $S$. Without using space-time diagrams, this is what the situation looks like.
In a space-time diagram, as drawn from the $S$ observer’s point of view:

Now from this picture, we will derive what this velocity $v$ is, in terms of $k$.

Let’s send a beam of light to observer $S'$, since $S'$ is holding a mirror, the light will be reflected back to us.

$k$ is the doppler effect. If observer $S$ had sent 2 light beams at time $t$ apart, the $S'$ observer would receive the 2 light beams a time $t'$ apart. The ratio $\frac{t'}{t}$ is called $k$.

Okay, so let’s find what $v$ is in terms of $k$.

What is the distance $d$?

That is the distance the mirror is away from observer $S$ when the light beam arrives.

Since light travels $c$ in both directions, $d = \frac{c(t_2 - t_1)}{2}$.

But $ct_2 = kr(t_1)$, so that gives us:

$d = \frac{c(k^2 - 1)t_1}{2}$.

Now, we use the concept of synchronized clocks to find out what time it was in the $S$ system when the light beam reached the mirror.

This would be the average of the two times $ct_2$ and $ct_1$.

So, $ct_m = \frac{ct_1 + ct_2}{2} = \frac{c(k^2 + 1)t_1}{2}$.
Now, since the mirror travelled a distance $d$ in a time $t_m$, we can find $v$.

$$v = \frac{d}{t_m} = \frac{c(k^2-1)x_l x_1}{2c(k^2+1)x_1} = \frac{c(k^2-1)}{(k^2+1)}$$

Then, what is $k$?

$$\frac{v}{c} = \frac{k^2-1}{k^2+1} \Rightarrow (\frac{v}{c})(k^2+1) = k^2-1 \Rightarrow k^2(1-\frac{v}{c}) = \frac{v}{c} + 1.$$

Or, $$k = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$$

Now, let's find out what $\gamma$, the time-dilation factor is.

In the $S'$ system, it was a time $k\tau$, when the light beam arrived.

This same event as observed in the $S$ system was $t_m$.

The ratio $t_m$ to $k\tau$, is $\gamma$.

$$\gamma = \frac{t_m}{k\tau} = \frac{(k^2+1)x_l}{2kx_1} = \frac{1+\frac{v}{c}}{2k} + 1 = \frac{1+\frac{v}{c}}{2k} \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}$$

$$= \sqrt{\frac{1}{1-\frac{v}{c}^2}} = \sqrt{1-\frac{v^2}{c^2}} \quad \text{O yay!}!!$$

Okay, so what, in general, is true about these spacetime diagrams?

First, it's very easy to show that 2 events that appear simultaneous in $S$, will not appear simultaneous in $S'$. 


From $s'$'s point of view, the two events $P$ and $Q$ are simultaneous.

However, in $(s')$'s point of view, events $P$ and $Q$ are different times.

At any one moment in space-time, there is a specific region where we could possibly end up. These regions take as their limits, the distance light could travel to. Since we cannot travel faster than the speed of light, we can only get to the "timelike" places.

Now, let's study the geometry of space-time in special relativity.

We must now show that the angle $ct'$ makes with $ct$ is the same as the angle $x'$ makes with $x$.

This is really an exercise in math ... so hopefully we can do it right and get it over with!

In the end, $\theta = \phi$. 
Since we are trying to establish the $x'$ axis, we will send beams of light to a mirror in the $s'$ system and record the time it took to get to the mirror and get back from the mirror. These times, of course, must be the same because light always travels the same speed in the same medium.

Now, in a space-time diagram, it looks like this: since the distance to mirror is $l$, that is $\overline{OB}$ in our picture. And since light took just as long getting there as coming back, $\overline{AO} = \overline{OB}$.

Now let's get down to business!!!
Next, let's study triangle BOD:

\[
\begin{align*}
\angle BOD &= 45^\circ. \\
\text{Why? Because} \\
180 - (45^\circ + 45^\circ) - (90^\circ - \theta) &= 45^\circ + \theta.
\end{align*}
\]

So that gives us two triangles. Now we use the Law of Sines.

\[
\frac{S}{\sin(45^\circ - \theta)} = \frac{l}{\sin(45^\circ - \theta)} \quad \text{and} \quad \frac{S}{\sin(45^\circ + \theta)} = \frac{l}{\sin(45^\circ + \theta)}.
\]

\[
\frac{S}{l} = \frac{\sin(45^\circ - \theta)}{\sin(45^\circ - \theta)} = \frac{\sin(45^\circ + \theta)}{\sin(45^\circ + \theta)}.
\]

Rearranging gives us: \( \frac{\sin(45^\circ - \theta)}{\sin(45^\circ + \theta)} = \frac{\sin(45^\circ)}{\sin(45^\circ + \theta)} \). The only way this statement can always be true is if \( \theta = \theta \).

Hence, the angles that \( x' \) makes with \( x \) equals the angle \( x' \) makes with \( x \).

In the case of light, this angle is \( 45^\circ \) and \( \theta \) is so big that it doesn't experience time. In other words, time stands still and the universe shrinks to a point.