Differential Form of Coulomb's Law

We have seen that Coulomb's Law can be written in integral form as:

\[ \text{Vol} \int \rho(r) \, dV = \oint_{\text{surf}} \mathbf{E} \cdot d\mathbf{A} \]

We now need to put this into a differential form. To do this we begin by considering the ways in which a vector can change. There are basically two ways: it can change along its direction or perpendicular to its direction. Since the first would have only one value (one direction) and the second two (two directions ± to the direction of the vector), we expect the first to be described by a scalar differential operation and the second by a vector operation. We begin with the first.

Divergence of a Vector

We define the divergence of a vector by:

\[ (\nabla \cdot \mathbf{P})_{AV} = \frac{1}{V} \oint_{\text{surf}} \mathbf{P} \cdot d\mathbf{A} \]
where \((\nabla \cdot \mathbf{A})_{\text{avg}}\) is the average value of the divergence of a vector \(\mathbf{A}\) over a closed volume \(V\), and \(\int_{\partial V} \mathbf{A} \cdot d\mathbf{A}\) is the integral over the bounding surface of the normal component of \(\mathbf{A}\). Just as for Gauss's Law this will give the net 
number leaving or entering the volume, and hence the change in \(\mathbf{A}\) along its direction. To see this consider a cube. The net number leaving the \(x\) faces will be the number leaving minus the number entering at \(x\) — hence the rate of change in \(\mathbf{A}\) in the \(x\) direction. It \(\mathbf{A}\) has only an \(x\) component. This will be the rate of change along its direction. Similarly for \(y\) and \(z\).

We then define divergence as:

\[
\text{div } \mathbf{A} = \lim_{V \to 0} \frac{1}{V} \int_{\partial V} \mathbf{A} \cdot d\mathbf{A}
\]

We can use this definition to calculate \(\text{div } \mathbf{A}\) in any coordinate system.
In Cartesian co-ordinates we consider a cubical box with corners at:

\[(x, y, z), (x+dx, y, z), (x, y+dy, z), (x+dx, y+dy, z), (x, y, z+dz), (x+dx, y, z+dz)\]

Then:

\[
du \bar{B} = \frac{1}{dx \, dy \, dz} \left[ \left( P(x+dx, y, z) - P_x(x, y, z) \right) dy \, dz \right.
\]

\[
+ \left\{ P_y(x, y+dy, z) - P_y(x, y, z) \right\} dx \, dz
\]

\[
+ \left\{ P_z(x, y, z+dz) - P_z(x, y, z) \right\} dx \, dy \right]
\]

\[
= \frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z} = \left( \frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z} \right)
\]

\[
= \bar{\nabla} \cdot \vec{P}
\]

For this reason we write \( du \bar{B} \) as \( \vec{\nabla} \cdot \vec{P} \).

In spherical co-ordinates we would consider the "box" formed by lines from

\[(r, \theta, \phi) \text{ to } (r+dr, \theta, \phi), \text{ from } (r, \theta+\delta \theta, \phi) \text{ to } (r+dr, \theta+\delta \theta, \phi), \]

\[(r+dr, \theta, \phi) \text{ to } (r+dr, \theta+\delta \theta, \phi), \text{ from } (r, \theta, \phi) \text{ to } (r, \theta+\delta \theta, \phi), \]

\[(r, \theta+\delta \theta, \phi) \text{ to } (r+dr, \theta+\delta \theta, \phi), \text{ from } (r+dr, \theta, \phi) \]
\[(r, \theta, \phi) \to (r, \theta, \phi + \delta \phi), \quad \text{from} \]
\[(r, \theta + \delta \theta, \phi) \to (r, \theta + \delta \theta, \phi + \delta \phi), \quad \text{etc.}\]

Then:

\[
d w \overline{P} = \frac{1}{4 \pi r^3 \sin \theta} \left[ P_n (r, \theta, \phi) - P_n (r + \delta r, \theta, \phi) \right] \\
+ \left\{ P_n (r, \theta + \delta \theta, \phi) \sin \delta \phi d \phi - P_n (r, \theta, \phi) \sin \delta \phi d \phi \right\} \\
+ \left\{ P_n (r, \theta, \phi + \delta \phi) \sin \delta \phi d \phi - P_n (r, \theta, \phi) \sin \delta \phi d \phi \right\}
\]

\[
= \frac{1}{r^2 \sin \theta} \left[ P_n (r, \theta, \phi) + \frac{\partial P_n}{\partial r} \right] (r \sin \theta \cos \phi \, d \phi d \theta d \phi) \\
+ 2 \, r \sin \theta \cos \phi \, d \phi d \theta d \phi - P_n (r, \theta, \phi) \sin \theta \cos \phi \, d \phi d \theta d \phi
\]

\[
+ \left\{ (P_n (r, \theta, \phi) + \frac{\partial P_n}{\partial \theta} \sin \theta \cos \phi \, d \phi d \theta - P_n (r, \theta, \phi) \sin \theta \cos \phi \, d \phi d \theta \right\}
\]

\[
+ \left\{ (P_n (r, \theta, \phi) + \frac{\partial P_n}{\partial \phi} \sin \theta \cos \phi \, d \phi d \theta - P_n (r, \theta, \phi) \sin \theta \cos \phi \, d \phi d \theta \right\}
\]

\[
= \frac{1}{r^2 \sin \theta} \left[ P_n (r, \theta, \phi) (2 \sin \theta \cos \phi \, d \phi d \theta + \sin \phi \, d \phi d \theta \right) \\
+ \frac{\partial P_n}{\partial r} (r \sin \theta \cos \phi \, d \phi d \theta + 2 \sin \theta \cos \phi \, d \phi d \theta + \sin \phi \, d \phi d \theta \right) \\
+ \frac{\partial P_n}{\partial \theta} (r \sin \theta \cos \phi \, d \phi d \theta + P_n (r, \theta, \phi) \cos \theta \, d \phi d \theta + \frac{\partial P_n}{\partial \phi} \sin \phi \, d \phi d \theta \right)
\]
Now take the limit as \( dr, d\theta, dq \to 0 \).

Then:

\[
\Delta n = \frac{2P_r}{r} + \frac{V_{\rho \theta}}{r} + \frac{1}{r} \frac{\partial P_r}{\partial \theta} + \frac{\partial (\theta P_\theta + \frac{1}{r^2} \rho \theta)}{\partial \theta} \frac{\Delta \theta}{\theta}
\]

But:

\[
\Delta n = \frac{1}{r} \frac{\partial n}{\partial r} + \frac{\theta}{r} \frac{\partial n}{\partial \theta} + \frac{\partial n}{\partial \theta} \frac{1}{\sin \theta} \frac{\Delta \theta}{\theta}
\]

Here:

\[
\Delta n = \left( \frac{1}{r} \frac{\partial n}{\partial r} + \frac{\theta}{r} \frac{\partial n}{\partial \theta} + \frac{\partial n}{\partial \theta} \frac{1}{\sin \theta} \frac{\Delta \theta}{\theta} \right) - \left( \frac{\partial n}{\partial r} + 2 \frac{\partial n}{\partial \theta} \frac{1}{\sin \theta} \frac{\Delta \theta}{\theta} \right)
\]

\[
= \frac{1}{r} \left[ \frac{\partial n}{\partial r} + \rho \frac{\partial n}{\partial \rho} + \frac{\partial n}{\partial \theta} + \frac{\partial n}{\partial \theta} \frac{1}{\sin \theta} \frac{\Delta \theta}{\theta} \right]
\]

\[
= \frac{\partial n}{\partial r} \left[ \frac{1}{r} \frac{\partial n}{\partial r} + \frac{\rho}{\partial \theta} \frac{\partial n}{\partial \rho} + \frac{\partial n}{\partial \theta} + \frac{\partial n}{\partial \theta} \frac{1}{\sin \theta} \frac{\Delta \theta}{\theta} \right]
\]

From last semester we know:

\[
\frac{\partial n}{\partial r} = \frac{\partial n}{\partial \rho} = \frac{\partial n}{\partial \theta} = 0
\]

\[
\frac{\partial n}{\partial \theta} = \theta, \quad \frac{\partial n}{\partial \phi} = -1, \quad \frac{\partial n}{\partial \phi} = 0
\]

\[
\frac{\partial n}{\partial \theta} = \sin \theta, \quad \frac{\partial n}{\partial \phi} = \cos \theta, \quad \frac{\partial n}{\partial \phi} = -\sin \theta \cos \theta
\]

Here:

...
\[ \vec{V} - \vec{P} = \vec{V} - \left[ \frac{\partial P_t}{\partial t} + \frac{\partial \Phi}{\partial t} + \frac{\partial}{\partial t} \left( \frac{\partial P_t}{\partial \phi} \right) \right] \]

\[ + \left[ \frac{\partial P_t}{\partial x} + \frac{\partial \Phi}{\partial x} + \frac{\partial}{\partial x} \left( \frac{\partial P_t}{\partial \phi} \right) \right] \]

\[ + \frac{1}{v \sin \theta} \left[ \frac{\partial P_t}{\partial \theta} + \frac{\partial \Phi}{\partial \theta} + \frac{\partial}{\partial \theta} \left( \frac{\partial P_t}{\partial \phi} \right) \right] \]

\[ + P \phi \left( \sin \theta - \cos \theta \right) \]

\[ = \frac{\partial P_t}{\partial t} + \frac{\partial \Phi}{\partial t} + \frac{\partial}{\partial t} \left( \frac{\partial P_t}{\partial \phi} \right) \]

\[ + \frac{\partial P_t}{\partial x} + \frac{\partial \Phi}{\partial x} + \frac{\partial}{\partial x} \left( \frac{\partial P_t}{\partial \phi} \right) \]

\[ + \frac{\partial P_t}{\partial \theta} + \frac{\partial \Phi}{\partial \theta} + \frac{\partial}{\partial \theta} \left( \frac{\partial P_t}{\partial \phi} \right) \]

\[ + P \phi \left( \sin \theta - \cos \theta \right) \]

Here again:

\[ d \omega \Phi = \vec{V} \cdot \vec{P} \]

This is true in all coordinate systems.

We now turn to the variation in \( \vec{P} \) in the direction of \( \vec{P} \). An easy way to visualize this is to consider flow of water in a stream in which the current decreases as you go deeper in the stream (remember the viscous flow problems of last semester). Treating the current as the vector \( \vec{P} \) the situation would look like this (as seen from the side).
The lines are closer together at the top cuz the magnitude of \( \vec{P} \) is larger. Now imagine placing a paddle wheel in the stream.

Clearly, the wheel will spin clockwise. The speed of the wheel is a measure of the variation of \( \vec{P} \) in its direction. We can get a value for this by calculating a line integral around a closed curve lying in the plane of the figure (curve C).
Clearly this is 0 if there is no variation in the $1$ component of $\vec{F}$. In our case it is non-zero with a value proportional to the variation in the $1$ component of $\vec{F}$. The sign depends on the direction we go around the curve. We adopt the convention of always keeping the interior of the curve to our left. Here we go counterclockwise as shown in the sketch. We then define the $1$ component of the average value of the curl of $\vec{F}$ to be:

$$\langle \vec{m} \cdot \text{curl} \vec{B} \rangle_{\text{surf}} = \frac{1}{S} \int_{\mathcal{S}} \text{curl} \vec{F} \cdot d\vec{r}$$

where $S$ is an arbitrary surface bounded by the curve $C$. The direction of $\vec{m}$ is given by the right hand rule: curl fingers of right hand in direction of $\vec{C}$ and thumb points in direction of $\vec{m}$. Now define a component of curl $\vec{F}$ by

$$\text{curl}(\vec{F}) = \lim_{S \to 0} \frac{1}{S} \int_{\mathcal{S}} \text{curl} \vec{F} \cdot d\vec{r}$$
Note that the surface $S$ is not a closed surface. It is the top half of an orange cut in half.

Again, we can evaluate this in any coordinate system. Using Cartesian coordinates, consider the $z$ component

\[
\mathbf{F} \cdot d\mathbf{r} = \mathbf{F}(x, y, z) \cdot \mathbf{r}_z = \mathbf{F}(x, y, z) \cdot (0, 0, dz) = F_z(x, y, z) \, dz
\]

where \( F_z(x, y, z) \) is the $z$ component of \( \mathbf{F}(x, y, z) \).

\[
= \int \left[ \frac{\partial F_z}{\partial x} \frac{\partial z}{\partial x} - \frac{\partial F_z}{\partial y} \frac{\partial z}{\partial y} \right] \, dx \, dy
\]
\[ \text{curl } \overrightarrow{P} = \frac{d}{dy} \left[ \frac{\partial P_y}{\partial x} - \frac{\partial P_x}{\partial y} \right] \]

Note that as \( S \to 0 \), the surface becomes a plane of area \( dx \, dy \). Thus

\[ \text{curl } \overrightarrow{P} = \frac{\partial P_y}{\partial x} - \frac{\partial P_x}{\partial y} \]

Similarly, we find:

\[ \text{curl } \overrightarrow{P}_x = \frac{\partial P_y}{\partial y} - \frac{\partial P_y}{\partial y} \]
\[ \text{curl } \overrightarrow{P}_y = -\frac{\partial P_x}{\partial x} + \frac{\partial P_x}{\partial x} \]

Now consider:

\[ \overrightarrow{G} \times \overrightarrow{B} = \left( \frac{\partial P_y}{\partial x} - \frac{\partial P_x}{\partial y} \right) \times \left( A_x \, \mathbf{i} + A_y \, \mathbf{j} + A_z \, \mathbf{k} \right) \]

\[ = \mathbf{i} \times \left( \frac{\partial P_y}{\partial x} \cdot \mathbf{j} + \frac{\partial P_x}{\partial y} \cdot \mathbf{k} + \frac{\partial P_y}{\partial y} \cdot \mathbf{i} + \frac{\partial P_x}{\partial x} \cdot \mathbf{j} + \frac{\partial P_y}{\partial y} \cdot \mathbf{i} + \frac{\partial P_x}{\partial x} \cdot \mathbf{j} \right) \]

\[ = \mathbf{i} \times \left( \frac{\partial P_y}{\partial x} \cdot \mathbf{j} + \frac{\partial P_x}{\partial y} \cdot \mathbf{k} \right) \]

\[ = \mathbf{i} \left( \frac{\partial P_y}{\partial x} - \frac{\partial P_x}{\partial x} \right) + \mathbf{j} \left( \frac{\partial P_y}{\partial y} - \frac{\partial P_x}{\partial y} \right) + \mathbf{k} \left( \frac{\partial P_y}{\partial y} - \frac{\partial P_x}{\partial y} \right) \]

\[ \text{curl } \overrightarrow{P} = \overrightarrow{G} \times \overrightarrow{B} \]

Again, we can do the same calculation on any vector field with the same
result.

Maxwell Equations for Electrostatics

The integral form of Gauss's Law is:

\[ \oint_{\text{Surf}} \mathbf{E} \cdot d\mathbf{A} = 4\pi \rho \text{ vol} \]

But \( \frac{1}{\epsilon_0} \oint_{\text{Surf}} \mathbf{E} \cdot d\mathbf{A} = \left( \mathbf{E} \cdot \mathbf{n} \right)_\text{Surf} = \frac{1}{\epsilon_0} \oint_{\text{Surf}} \mathbf{E} \cdot d\mathbf{A} \]

\[ \oint_{\text{Surf}} \mathbf{E} \cdot d\mathbf{A} = \oint_{\text{Vol}} \mathbf{E} \cdot d\mathbf{A} = 4\pi \rho \text{ vol} \]

Since this must hold for any volume we have:

\[ \mathbf{n} \cdot \mathbf{E} = \frac{4\pi \rho}{\epsilon_0} = \frac{\Phi}{\epsilon_0} \]

where \( \mathbf{n} \cdot \mathbf{E} = \frac{1}{4\pi \epsilon_0} \)

This is the 1st Maxwell Equation, which is just the differential form of Coulomb's Law.

We can now answer another important question about electrostatics - is Coulomb's Law a conservative force? We recall that a conservative force is one for...
which the work done is independent of the path — depends only on the endpoints. We have:

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_A^B \mathbf{E} \cdot d\mathbf{r} + \int_B^A \mathbf{E} \cdot d\mathbf{r}
\]

But \( \int \mathbf{A} \cdot d\mathbf{r} = \frac{1}{2} \int \mathbf{E} \cdot d\mathbf{r} \) for some suitable \( \mathbf{A} \). Hence \( \mathbf{E} \) is conservative if \( \nabla \times \mathbf{E} = 0 \).

Because at separation, if the field of a single charge is conservative so is the field of any # of them. Hence we need:

\[
\nabla \times \mathbf{E} = 0
\]
\[
\hat{\mathbf{A}} = \left( \frac{\hat{r}}{c^2} + \frac{\hat{\theta}}{r} + \frac{\hat{\phi}}{\rho \sin \phi} \right) \times \frac{\hat{\mathbf{B}}}{\rho^2}
\]

\[
= r \times \left[ -\frac{2 \hat{\mathbf{B}}}{r^2} + \frac{\hat{\mathbf{B}}}{r^2} \frac{\mathbf{r}}{r} \right]
\]

\[
+ \frac{\hat{\mathbf{r}}}{r} \left[ \frac{\hat{\mathbf{B}}}{r^2} \frac{\mathbf{r}}{r} \right] + \hat{\mathbf{r}} \times \left[ \frac{\hat{\mathbf{B}}}{r^2} \frac{\mathbf{r}}{r} \right]
\]

\[
= \frac{\hat{\mathbf{B}}}{r^2} \hat{\mathbf{r}} \times \hat{\mathbf{r}} + \frac{\hat{\mathbf{B}}}{r^2} \hat{\mathbf{r}} \times \sin \phi \hat{\phi} = 0
\]

Here \( \hat{\mathbf{r}} \times \hat{\mathbf{r}} = 0 \) and \( \hat{\mathbf{E}} \) is conservative. This will be made clear later when we come to non-stationary fields.