We now consider a special charge configuration which will have importance later on. Imagine two equal and opposite point charges placed very close together as indicated in the following sketch

![Diagram of two charges](image)

We define a point dipole to be this configuration in the limit

$$\vec{p} = \lim_{q \to \infty} q\vec{\ell}$$

where \( p \) is the dipole moment. Although it might seem that such an arrangement is purely artificial and could never be found in practice, it turns out to have a fundamental importance which will become apparent later on in both electrostatics and magnetism.

We now consider the electric field produced by such an arrangement. There are several ways to do this. The most obvious is to simply add up the fields produced by each charge separately. We begin by finding the field along the axis of the dipole

$$\vec{E} = \hat{x} \left[ \frac{kq}{(r - \ell/2)^2} - \frac{kq}{(r + \ell/2)^2} \right] = \hat{x} \frac{kq}{r^2} \left[ \frac{1}{\left(1 - \frac{\ell}{2r}\right)^2} - \frac{1}{\left(1 + \frac{\ell}{2r}\right)^2} \right]$$

$$= \frac{kq}{r^2} \left[ \frac{\left(1 + \frac{\ell}{2r}\right)^2 - \left(1 - \frac{\ell}{2r}\right)^2}{\left(1 - \frac{\ell^2}{4r^2}\right)} \right] = kq \frac{2\ell}{r^2} \left[ \frac{2\ell}{\left(1 - \frac{\ell^2}{4r^2}\right)} \right] = \frac{2kp}{r^3 \left(1 - \frac{\ell^2}{4r^2}\right)^2}$$

We now take the limit \( q \to 0, \ \ell \to \infty \) to get
\[ \vec{E} = \hat{x} \frac{2k\rho}{r^3} = \frac{2k\hat{\rho}}{r^3} \]

Next consider the field at directly above the dipole.

\[ \vec{E} = -\hat{x} kq \left[ \frac{1}{\left( r^2 + \frac{\ell^2}{4} \right)^{3/2}} \right] = -\hat{x} \frac{kp\ell}{\left( r^2 + \frac{\ell^2}{4} \right)^{3/2}} = -\hat{x} \frac{kp}{\left( r^2 + \frac{\ell^2}{4} \right)^{3/2}} \]

Now take the limit as \( \ell \to 0 \)

\[ \vec{E} = -\hat{x} \frac{kp}{r^3} = -\frac{kp}{r^3} \]

In general the geometry is more complicated and the result is easiest found by doing some vector calculus as follows. What we really have is the field due to two point charges which we can write as

\[ \vec{E}(\vec{r}) = \vec{E} \left( \vec{r} - \frac{\ell}{2} \right) - \vec{E} \left( \vec{r} + \frac{\ell}{2} \right) \]
where

\[ E(\hat{r}) = \frac{kq\hat{r}}{r^2} \]

Then this looks like a derivative

\[ \vec{E}_x\left(\vec{r} - \frac{\ell}{2}\right) = E_x\left(x - \frac{\ell x}{2}, y - \frac{\ell y}{2}, z - \frac{\ell z}{2}\right) \]

\[ = E_x(x, y, z) - \frac{\ell_x}{2} \frac{\partial E_x}{\partial x} - \frac{\ell_y}{2} \frac{\partial E_x}{\partial y} - \frac{\ell_z}{2} \frac{\partial E_x}{\partial z} \]

\[ = E_x(x, y, z) - \vec{\nabla}E_x \cdot \frac{\ell}{2} \]

Then

\[ \vec{E}(\vec{r}) = \vec{E}(\vec{r}) - \left(\vec{\nabla}E_x\right) \cdot \frac{\ell}{2} - \left(\vec{\nabla}E_y\right) \cdot \frac{\ell}{2} - \left(\vec{\nabla}E_z\right) \cdot \frac{\ell}{2} - \vec{E}(\vec{r}) - \left(\vec{\nabla}E_x\right) \cdot \frac{\ell}{2} - \left(\vec{\nabla}E_y\right) \cdot \frac{\ell}{2} - \left(\vec{\nabla}E_z\right) \cdot \frac{\ell}{2} \]

\[ = -\left(\vec{\nabla}E_x\right) \cdot \ell - \left(\vec{\nabla}E_y\right) \cdot \ell - \left(\vec{\nabla}E_z\right) \cdot \ell \]

But

\[ \vec{\nabla}E_x = \vec{\nabla} \frac{kq}{(x^2 + y^2 + z^2)^{3/2}} = kq \left[ \hat{x} \left( \frac{1}{r^3} - \frac{3x^2}{r^5} \right) + \hat{y} \left( -\frac{3xy}{r^5} \right) + \left( -\frac{3xz}{r^5} \right) \right] \]

\[ \therefore \left(\vec{\nabla}E_x\right) \cdot \ell = kq \left[ \frac{\ell_x}{r^2} - \frac{3x}{r} \right] \]

and

\[ \vec{E}(\vec{r}) = \frac{k}{r^3} \left[ 3(\vec{p} \cdot \hat{r})\hat{r} - \vec{p} \right] \]

Check this in the two cases we have worked out

Case 1: \( \vec{r} = r\hat{p} \)

\[ \vec{E}(\vec{r}) = \frac{k}{r^3} \left[ 3p\hat{r} - p\hat{r} \right] = \frac{2kp\hat{p}}{r^3} \]

as before.
Case 2: \( \vec{r} \perp \vec{p} \)

\[ \vec{E}(\vec{r}) = \frac{k}{r^3} [-\vec{p}] = -\frac{k\vec{p}}{r^3} \]

as before.