We now consider a particularly important charge configuration—a dipole. This consists of two equal but opposite charges separated by a small distance.

We define the dipole moment as

\[
\mathbf{p} = \lim_{q \to \infty, \ell \to 0, q \ell = \text{const}} q \ell
\]

Although it might seem that such an entity would not be encountered in practice, it has a fundamental importance as seen by considering a localized charge distribution.

The voltage produced by such a charge distribution is given by

\[
V(\mathbf{r}) = k \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3 r'
\]

If we are far away from the distribution we have \( r >> r' \) and can expand the denominator in a multi-dimensional Taylor series.

To see how this works, consider the 2-dimensional case. We expand about the point \((0,0)\). Then to find the value at \((\Delta x, \Delta y)\) we follow the path shown.

Recalling the definition of a derivative

\[
\lim_{dx \to 0} \frac{f(x + dx) - f(x)}{dx} = \frac{df}{dx}
\]
we get

\[ f(\Delta x,0) = f(0,0) + \int_0^{\Delta x} \frac{df(x,0)}{dx} \, dx \]

Hence, we need \(df(x,0)/dx\). But this is

\[ \frac{df(x,0)}{dx} = \frac{df}{dx}\bigg|_{0,0} + \int_0^x \frac{d^2f(x,0)}{dx^2} \, dx \]

But

\[ \frac{d^2f(\epsilon,0)}{dx^2} = \frac{d^2f}{dx^2}\bigg|_{0,0} + \int_0^\epsilon \frac{d^3f(x,0)}{dx^3} \, dx \]

We continue this process to whatever order we want. To third order it gives

\[ \frac{d^3f(\epsilon,0)}{dx^3} = \frac{d^3f}{dx^3}\bigg|_{0,0} + \int_0^\epsilon \frac{d^4f(x,0)}{dx^4} \, dx \]

\[ \frac{d^2f(x,0)}{dx^2} = \frac{d^2f}{dx^2}\bigg|_{0,0} + \int_0^\epsilon \frac{d^3f(x,0)}{dx^3} \, dx \]

\[ \frac{df(x,0)}{dx} = \frac{df}{dx}\bigg|_{0,0} + \left[ \frac{d^2f}{dx^2}\bigg|_{0,0} + \frac{d^3f}{dx^3}\bigg|_{0,0} \right] \epsilon + \frac{x^2}{2} + \ldots \]

\[ f(\Delta x,0) = f(0,0) + \left[ \frac{df}{dx}\bigg|_{0,0} + \frac{1}{2} \frac{d^2f}{dx^2}\bigg|_{0,0} \right] \Delta x + \frac{1}{3} \frac{d^3f}{dx^3}\bigg|_{0,0} \left( \frac{\Delta x}{2} \right)^2 \]
Now we must go from \((\Delta x,0)\) to \((\Delta x,\Delta y)\). We do it in the same way getting

\[
f(\Delta x,\Delta y) = f(0,0) + \frac{\partial f}{\partial x} \bigg|_{0,0} \Delta x + \frac{1}{2!} \frac{\partial^2 f}{\partial x^2} \bigg|_{0,0} (\Delta x)^2 + \frac{1}{3!} \frac{\partial^3 f}{\partial x^3} \bigg|_{0,0} (\Delta x)^3 + \ldots + \int_0^{\Delta y} \frac{\partial f(\Delta x, y)}{\partial y} dy
\]

\[
= f(\Delta x,0) + \frac{\partial f}{\partial y} \bigg|_{\Delta x,0} \Delta y + \frac{1}{2!} \frac{\partial^2 f}{\partial y^2} \bigg|_{\Delta x,0} (\Delta y)^2 + \frac{1}{3!} \frac{\partial^3 f}{\partial y^3} \bigg|_{\Delta x,0} (\Delta y)^3 + \ldots
\]

Hence we need

\[
\frac{\partial f}{\partial y} \bigg|_{\Delta x,0} = \frac{\partial f}{\partial y} \bigg|_{0,0} + \frac{\Delta x}{0,0} + \frac{\partial^2 f}{\partial y \partial x} \bigg|_{\varepsilon,0} d\varepsilon = \frac{\partial f}{\partial y} \bigg|_{0,0} + \frac{\Delta x}{0,0} + \frac{\varepsilon}{0,0} + \frac{\partial^3 f}{\partial y \partial x^2} \bigg|_{\eta,0} d\eta \bigg|_{\varepsilon,0}
\]

To 3rd order this is

\[
\frac{\partial f}{\partial y} \bigg|_{0,0} + \left[ \frac{\partial^2 f}{\partial y \partial x} \bigg|_{0,0} + \frac{\varepsilon}{0,0} + \frac{\partial^3 f}{\partial y \partial x^2} \bigg|_{0,0} \right] d\varepsilon = \frac{\partial f}{\partial y} \bigg|_{0,0} + \Delta x \frac{\partial^2 f}{\partial y \partial x} \bigg|_{0,0} + \frac{1}{2!} \frac{\partial^3 f}{\partial y \partial x^2} \bigg|_{0,0} (\Delta x)^2
\]

Let us now keep only quadratic terms in \(\Delta x\) \(\Delta y\). Then for

\[
\frac{\partial^2 f}{\partial y^2} \bigg|_{\Delta x,0}
\]

We can use

\[
\frac{\partial^2 f}{\partial y \partial x} \bigg|_{0,0}
\]

Putting it all together we get

\[
f(\Delta x\Delta y) = f(0,0) + \frac{\partial f}{\partial x} \bigg|_{0,0} \Delta x + \frac{1}{2!} \frac{\partial^2 f}{\partial x^2} (\Delta x)^2 + \frac{\partial f}{\partial y} \bigg|_{0,0} \Delta y + \frac{\partial^2 f}{\partial y \partial x} \bigg|_{0,0} (\Delta x\Delta y) + \frac{1}{2!} \frac{\partial^2 f}{\partial y^2} \bigg|_{0,0} (\Delta y)^2 + \ldots
\]
Generalizing to 3 dimensions we find

\[ f(x, y, z) = f(0, 0, 0) + \frac{\partial f}{\partial x} \bigg|_{0,0,0} x + \frac{\partial^2 f}{\partial x^2} \bigg|_{0,0,0} \frac{x^2}{2!} + \frac{\partial f}{\partial y} \bigg|_{0,0,0} y + \frac{\partial^2 f}{\partial y^2} \bigg|_{0,0,0} \frac{y^2}{2!} + \]

\[ + \frac{\partial f}{\partial z} \bigg|_{0,0,0} z + \frac{\partial^2 f}{\partial z^2} \bigg|_{0,0,0} \frac{z^2}{2!} + \frac{\partial f}{\partial x \partial y} \bigg|_{0,0,0} xy + \frac{\partial^2 f}{\partial x \partial z} \bigg|_{0,0,0} xz + \frac{\partial^2 f}{\partial y \partial z} \bigg|_{0,0,0} yz + ... \]

Cultural Digression

If will be useful to consider quantities beyond scalars and vectors. In particular, we will need entities with two or more directions. In general, such quantities will be tensors of various ranks. In particular, we will need 2\(^{nd}\) rank tensors, which have 2 directions and, in 3 dimensions, 9 components. In 4 dimensions (case of relativity), they will have 16 components.

For now we will consider a specially simple case which we will write as \( \vec{A}\vec{B} \), a product of 2 vectors with neither a dot or a cross. This product is defined by its action on a 3\(^{rd}\) vector, \( \vec{C} \). We take

\[ \vec{A}\vec{B} \cdot \vec{C} \equiv \vec{A}(\vec{B} \cdot \vec{C}) \]

\[ \vec{C} \cdot \vec{A}\vec{B} \equiv (\vec{C} \cdot \vec{A})\vec{B} \]

\[ \vec{A}\vec{B} \times \vec{C} \equiv \vec{A}(\vec{B} \times \vec{C}) \]

\[ \vec{C} \times \vec{A}\vec{B} \equiv (\vec{C} \times \vec{A})\vec{B} \]

We also define a new product by

\[ \vec{A}\vec{B} : \vec{C}\vec{D} \equiv \vec{A}(\vec{B} \cdot \vec{C}) \cdot \vec{D} = (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C}) \]

Back to Taylor Series

We can write our result as

\[ f(\vec{r}) = f(\vec{0}) + \vec{r} \cdot [\vec{\nabla} f]_0 + \frac{1}{2} \vec{r} \cdot [\vec{\nabla}^2 f]_0 + ... \]

To see that this works we evaluate the 3\(^{rd}\) term
\[ \ddot{\nabla} \ddot{\nabla} f = \ddot{\nabla} \left[ x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} \right] = \left( \dot{x} \frac{\partial}{\partial x} + \dot{y} \frac{\partial}{\partial y} + \dot{z} \frac{\partial}{\partial z} \right) \left( \dot{x} \frac{\partial f}{\partial x} + \dot{y} \frac{\partial f}{\partial y} + \dot{z} \frac{\partial f}{\partial z} \right) \]
\[ = \dot{x} \ddot{x} \frac{\partial^2 f}{\partial x^2} + \dot{x} \dot{y} \frac{\partial^2 f}{\partial x \partial y} + \dot{x} \dot{z} \frac{\partial^2 f}{\partial x \partial z} + \dot{y} \dot{y} \frac{\partial^2 f}{\partial y^2} + \dot{y} \dot{z} \frac{\partial^2 f}{\partial y \partial z} + \dot{z} \dot{z} \frac{\partial^2 f}{\partial z^2} \]
\[ \ddot{r} : \ddot{\nabla} \ddot{f} = (\dot{x}x + \dot{y}y + \dot{z}z)(\ddot{x}x + \ddot{y}y + \ddot{z}z) : \left[ \begin{array}{c}
\ddot{x} \frac{\partial^2 f}{\partial x^2} + \ddot{y} \frac{\partial^2 f}{\partial y^2} + \ddot{z} \frac{\partial^2 f}{\partial z^2} \\
\ddot{y} \frac{\partial^2 f}{\partial x \partial y} + \ddot{z} \frac{\partial^2 f}{\partial x \partial z} + \ddot{x} \frac{\partial^2 f}{\partial y \partial x} + \ddot{x} \frac{\partial^2 f}{\partial y \partial x}
\end{array} \right] \]
\[ = x^2 \frac{\partial^2 f}{\partial x^2} + y^2 \frac{\partial^2 f}{\partial y^2} + z^2 \frac{\partial^2 f}{\partial z^2} + 2 \frac{\partial^2 f}{\partial x \partial y}xy + 2 \frac{\partial^2 f}{\partial x \partial z}xz + 2 \frac{\partial^2 f}{\partial y \partial z}yz \]
\[ \therefore \frac{1}{2} \ddot{r} : [\ddot{\nabla} \ddot{f}]_0 = \frac{1}{2!} \left( \frac{\partial^2 f}{\partial x^2} \right)_0 x^2 + \frac{\partial^2 f}{\partial y^2} \left|_0 \right. y^2 + \frac{\partial^2 f}{\partial z^2} \left|_0 \right. z^2 + \frac{\partial^2 f}{\partial x \partial y} \left|_0 \right. xy + \frac{\partial^2 f}{\partial x \partial z} \left|_0 \right. xz + \frac{\partial^2 f}{\partial y \partial z} \left|_0 \right. yz \]

Just as required.

Now we are in position to expand \(1/|\vec{r} - \vec{r}'|\) for the region \(r >> r'\). We get

\[ \frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{r} + \left( \frac{1}{2} \right) \ddot{\nabla} \ddot{\nabla} \frac{1}{|\vec{r} - \vec{r}'|} + \frac{1}{2} \ddot{r} : [\ddot{\nabla} \ddot{f}] \frac{1}{|\vec{r} - \vec{r}'|} \]

Consider the 2\textsuperscript{nd} term

\[ \ddot{\nabla} \frac{1}{|\vec{r} - \vec{r}'|} = \left( \dot{x} \frac{\partial}{\partial x} + \dot{y} \frac{\partial}{\partial y} + \dot{z} \frac{\partial}{\partial z} \right) \frac{1}{(x-x')^2 + (y-y')^2 + (z-z')^2}^{1/2} \]
\[ = \dot{x} \left( \frac{-1}{x} \right) \frac{1}{|\vec{r} - \vec{r}'|^3} + \dot{y} \frac{y-y'}{|\vec{r} - \vec{r}'|^3} + \dot{z} \frac{z-z'}{|\vec{r} - \vec{r}'|^3} = \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \]

\[ \therefore \ddot{\nabla} \frac{1}{|\vec{r} - \vec{r}'|} \bigg|_{r'=0} = \frac{\vec{r}}{r^3} \]

Hence the 2\textsuperscript{nd} term is
Now consider the 3rd term. We need
\[
\nabla' \nabla', \frac{1}{|\mathbf{r} - \mathbf{r}'|} = \nabla', \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} = \frac{-\nabla' \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} + \frac{3(\mathbf{r} - \mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^5}
\]

Now consider \( \nabla \mathbf{r} \). It is
\[
\nabla \mathbf{r} = \left( \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right)(x \hat{x} + y \hat{y} + z \hat{z}) = \hat{x}x + \hat{y}y + \hat{z}z = \hat{n}
\]

Hence,
\[
\nabla' \nabla', \frac{1}{|\mathbf{r} - \mathbf{r}'|} \bigg|_{r''=0} = \frac{3 \hat{x} \mathbf{r} - \hat{n}}{r^3}
\]

Hence, the 3rd term is
\[
\frac{1}{2} \mathbf{r} \cdot \mathbf{r}' = \frac{3 \hat{x} \mathbf{r} - \hat{n}}{r^3} \sim \frac{1}{r^3}
\]

Thus, each term in the series falls off as 1 higher power of \( r \). Thus at long distances only the lowest non-vanishing order will matter.

We have
\[
V(\mathbf{r}) = k \int \rho(\mathbf{r}') \left[ \frac{1}{r} + \frac{\hat{r} \cdot \mathbf{r}'}{r^2} + \frac{3 \hat{r} \mathbf{r} - \hat{n}}{r^3} + \ldots \right] \, d^3 r' = k \left[ \frac{Q}{r} + \frac{\hat{r}}{r^2} \cdot \int \rho(\mathbf{r}') \, d^3 r' + O \left( \frac{1}{r^3} \right) \right]
\]

If \( Q \neq 0 \) the 1st term normally dominates. But if \( Q = 0 \) then the 2nd or “dipole” term dominates, etc. We define the dipole moment of a charge distribution to be
\[
\mathbf{\tilde{p}} = \int \rho(\mathbf{r}') \hat{r}' \, d^3 \mathbf{r}'
\]

where the integral is over the volume of the charge distribution. Recall \( \mathbf{r} \) lies outside the distribution.

Note that if \( Q = 0 \) then \( \mathbf{\tilde{p}} \) is independent of the origin used. To see this let \( \mathbf{R}_0 \) locate the origin, \( \mathbf{R} \) be the vector from \( \mathbf{R}_0 \) to the charge.
Then $r' = \vec{R}_0 + \vec{R}$ and

$$p_0 = \int \rho(r')(\vec{R}_0 + \vec{R}) d^3r' = QR\vec{R}_0 + \int \rho(\vec{R}) d^3R = Q\vec{R}_0 + \vec{p} = \vec{p}$$

Hence $p_0$ is the same in either coordinate system (origin at $O$ or at $\vec{R}_0$). If $Q \neq O$ this is not true.

Next we compare this definition with the previous one in terms of point charges.

$$p = \int \rho(r')q d^3r' = -q\vec{R}_0 + q(\vec{R}_0 + \ell) = q\ell$$

just as before.

**Field Due to a Dipole**

We have

$$V(\vec{r}) = \frac{k\vec{r} \cdot \vec{p}}{r^2} = \frac{k\vec{r} \cdot \vec{p}}{r^3}$$

$$E(\vec{r}) = -\vec{\nabla}V(\vec{r}) = -k\left[\frac{\vec{V}(\vec{r} \cdot \vec{p})}{r^3} + (\vec{r} \cdot \vec{p})\vec{\nabla} \frac{1}{r^3}\right]$$
Hence, we need \( \vec{\nabla}(\vec{r} \cdot \vec{p}) \).

**Math Diversion**

We will need many vector calculus identities, and this is a good place to see how to get them. For the moment the ones of interest to us are those involving \( \vec{\nabla} \). They can all be obtained by noting the following facts and conventions.

1. \( \vec{\nabla} \) is a vector and obeys the usual rules of vector algebra.
2. \( \vec{\nabla} \) is a differential operator and obeys the differential rules, namely \( d(AB) = (dA)B + A(dB) \).
3. \( \vec{\nabla} \) is always written to left of the entity it is operating on.

We need the following combinations

1. \( \vec{\nabla}(fg) \)
2. \( \vec{\nabla} \times (\vec{A} \times \vec{B}) \)
3. \( \vec{\nabla} \cdot (\vec{A}a) \)
4. \( \vec{\nabla} \cdot (\vec{A} \times \vec{B}) \)
5. \( \vec{\nabla} \times (a\vec{A}) \)
6. \( \vec{\nabla}(\vec{A} \cdot \vec{B}) \)

We now do each of these in turn.

1. Use differential property
   \[
   \vec{\nabla}(fg) = (\vec{\nabla}f)g + f(\vec{\nabla}g)
   \]

3. Use both vector and differential properties
   \[
   \vec{\nabla} \cdot (\vec{A}a) = a\vec{\nabla} \cdot \vec{A} + \vec{A} \cdot (\vec{\nabla}a)
   \]

Note that \( \vec{\nabla} \) operates on both \( \vec{A} \) and \( a \), is always dotted with \( \vec{A} \), and always to the left of what it operates on.

2. Here, we use the triple product identity
   \[
   \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{c} \cdot \vec{a})\vec{b} - (\vec{b} \cdot \vec{a})\vec{c}
   \]

Let \( \vec{a} = \vec{\nabla}, \vec{b} = \vec{A}, \vec{c} = \vec{B} \). Operate first on \( \vec{A} \). Then
\( \hat{\nabla}_A \times (\hat{\nabla} \times \vec{B}) = (\hat{\nabla} \cdot \vec{B}) \vec{A} - (\hat{\nabla} \cdot \vec{A}) \vec{B} \)

(note that \( \hat{\nabla} \) is left of what it operates on). Now operate on \( \vec{B} \). For this, reverse the order of \( \vec{A} \) and \( \vec{B} \) in the cross product. This gives a minus sign. Then just interchange \( \vec{A} \) and \( \vec{B} \) in the above result

\( \hat{\nabla} \times (\vec{A} \times \vec{B}) = (\hat{\nabla} \cdot \vec{B}) \vec{A} - (\hat{\nabla} \cdot \vec{A}) \vec{B} - \vec{A} (\hat{\nabla} \cdot \vec{B}) + \vec{A} (\hat{\nabla} \cdot \vec{B}) \)

4. Here we use the fact that

\[
 a \cdot (\vec{b} \times \vec{c}) = b \cdot (\vec{c} \times \vec{a}) = c \cdot (\vec{a} \times \vec{b})
\]

Operate first on \( \vec{A} \). Let \( \vec{V} = \vec{a} \), \( \vec{A} = \vec{b} \), \( \vec{B} = \vec{c} \). Then

\[
 \hat{\nabla}_A \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{V} \times \vec{A})
\]

Now operate on \( \vec{B} \)

\[
 \hat{\nabla}_B \cdot (\vec{A} \times \vec{B}) = -\vec{A} \cdot (\vec{V} \times \vec{B})
\]

And to reverse \( \vec{C} \) and \( \vec{a} \) in the cross product, - sign. Thus

\[
 \vec{V} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{V} \times \vec{A}) - \vec{A} \cdot (\vec{V} \times \vec{B})
\]

5. Here we use differential operator rule and keep \( \vec{V} \) crossed into \( \vec{A} \). Thus

\[
 \hat{\nabla} \times (a\vec{A}) = (\vec{V}a) \times \vec{A} + a(\vec{V} \times \vec{A})
\]

6. Use the triple product again. Act first on \( \vec{A} \). Let \( \vec{V} = \vec{c} \), \( \vec{A} = \vec{b} \), \( \vec{B} = \vec{a} \). Then

\[
 \hat{\nabla}_A (\vec{A} \cdot \vec{B}) = (\vec{B} \cdot \vec{V}) \vec{A} + \vec{B} \times (\vec{V} \times \vec{A})
\]

Since \( \vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A} \) we get

\[
 \hat{\nabla}_B (\vec{A} \cdot \vec{B}) = (\vec{A} \cdot \vec{V}) \vec{B} + \vec{A} \times (\vec{V} \times \vec{B})
\]

Hence

\[
 \hat{\nabla}(\vec{A} \cdot \vec{B}) = (\vec{B} \cdot \vec{V}) \vec{A} + \vec{B} \times (\vec{V} \times \vec{A}) + (\vec{A} \cdot \vec{V}) \vec{B} + \vec{A} \times (\vec{V} \times \vec{B})
\]
Any vector calculus operation can be derived in similar fashion following the rules above.

I strongly recommend you learn how to do this rather than just memorize the results. You can always look them up, but that is a very poor way to do it.