ENERGY CONSERVATION IN ELECTROMAGNETIC FIELDS

We now look at energy conservation in electromagnetic fields. We begin by considering the work done on moving charges by the fields. We have:

\[
\text{dW} = \vec{F} \cdot d\vec{r} = q(\vec{E} + \vec{v} \times \vec{B}) \cdot d\vec{r}
\]

Then

\[
\frac{\text{dW}}{dt} = P = q(\vec{E} + \vec{v} \times \vec{B}) \cdot \frac{d\vec{r}}{dt} = q\vec{E} \cdot \vec{v} + q(\vec{v} \times \vec{B}) \cdot \vec{v}
\]

But \( \vec{v} \times \vec{B} \) is perpendicular to \( \vec{v} \). Hence

\[
(\vec{v} \times \vec{B}) \cdot \vec{v} = 0
\]

Thus static \( \vec{B} \) fields do no work! Does this mean that \( \vec{B} \) fields never do work? No. A time dependent \( \vec{B} \) field will produce an electric field (by Faraday’s law) and the \( \vec{E} \) field will generally do work. If we now have a collection of charges we get:

\[
P = \sum_{i} q_{i} \vec{E} \cdot \vec{v}_{i}
\]

In the continuum limit this becomes:

\[
P = \int \vec{E} \cdot \vec{J} \, d\text{vol}
\]

(Recall that \( qv \rightarrow \text{Idl} \rightarrow k\text{dA} \rightarrow J\text{dvol} \) as the number of dimensions increases.) Hence the work/sec done by the fields in a volume \( V \) is:

\[
P = \int_{V} \vec{E} \cdot \vec{J} \, d\text{vol}
\]

We now use the Maxwell equation:

\[
\vec{v} \times \vec{H} = \vec{J} + \epsilon \frac{\partial \vec{E}}{\partial t} \rightarrow \vec{J} = \vec{v} \times \vec{H} - \frac{\partial \vec{D}}{\partial t}
\]

to get

\[
\int_{V} \vec{E} \cdot \vec{J} \, d\text{vol} = \int_{V} \vec{E} \left[ \vec{v} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} \right] \, d\text{vol}
\]
But

\[ \nabla \cdot (\vec{E} \times \vec{H}) = (\nabla \times \vec{E}) \cdot \vec{H} - \vec{E} \cdot (\nabla \times \vec{H}) \]

\[ \therefore \int_{\text{vol}} \vec{E} \cdot \vec{J} \, d\text{vol} = \int_{\text{vol}} \left[ \vec{H} \cdot (\nabla \times \vec{E}) - \nabla \cdot (\vec{E} \times \vec{H}) - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \right] \, d\text{vol} \]

\[ = \int_{\text{vol}} \left[ -\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \right] \, d\text{vol} - \int_{\text{surf}} (\vec{E} \times \vec{H}) \cdot d\vec{S} \]

We know that

\[ \int \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \, d\text{vol} \]

is the rate at which energy is going into the electric field. Similarly:

\[ \int \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \, d\text{vol} \]

is the rate at which energy is going into the magnetic field. Thus:

\[ \int_{\text{vol}} \left[ \vec{E} \cdot \vec{J} + \frac{dU_{\text{EM}}}{\partial t} \right] \, d\text{vol} = -\int_{\text{surf}} (\vec{E} \times \vec{H}) \cdot d\vec{S} \]

where \( U_{\text{EM}} \) is the electromagnetic field energy/volume. The left hand side is then the rate at which energy is appearing in the volume \( V \). Conservation of energy then requires that the right hand side be the rate at which energy is entering the volume. Thus we can interpret:

\[ \vec{S} = \vec{E} \times \vec{H} \]

as the energy/sec/area carried by the electromagnetic field. \( \vec{S} \) is called the “Poynting Vector”.

As an example of its use consider a current \( I \) flowing through a resistor \( R \).
We know

\[ I = \frac{V_{\text{in}} - V_{\text{out}}}{R} = \frac{\dot{E}L}{R} \]

The current will produce a magnetic field:

\[ \vec{B} = \frac{\mu_0 I}{2\pi r} \hat{\theta} \]

directed as shown. Then:

\[ \vec{S} = \vec{E} \times \frac{\vec{B}}{\mu_0} = \frac{RI}{L} \hat{z} \times \frac{\mu_0 I}{2\pi\mu_0} \hat{\theta} = -\hat{r} \frac{RI^2}{2\pi L} \]

Thus the energy entering the resistor/sec is:

\[ S2\pi rL = I^2 R \]

as expected!