CURL OF A VECTOR

We now turn to the question of how the vector changes in the other two directions. Graphically the two cases are as indicated in the following sketches:

\[ \frac{dB}{dx} \neq 0 \]

\[ \frac{dB}{dy} \neq 0 \]

We have just finished analyzing the top case – which was described by the divergence of the vector. We now consider the bottom case. As a physical model of the situation you can consider water flowing in a stream. The velocity increases as you get farther from the bottom (zero at the bottom of the stream). You could measure this effect by putting a paddle wheel in the stream:

Clearly the wheel will rotate cw as shown. We can get a measure of this effect by calculating the line integral of \( \mathbf{v} \cdot d\mathbf{\ell} \) around a closed loop:
On the top half of the loop the integral will be negative since \( \vec{v} \) and \( d\vec{\ell} \) are in opposite directions, while on the bottom half it will be positive. If:

\[
\frac{d\vec{v}}{dy} = 0
\]

the two would cancel, otherwise they in general will not. With this as a guide we proceed as follows.

Consider an arbitrary closed curve. Now construct an open surface ending on the curve and lying in the region on one side of the curve. Imagine slicing an orange in half. The cut is the curve and the skin of one half of the orange is the surface. To decide which half, go around the curve keeping it to your left. Then curl the fingers of your right hand in the direction you are going and your thumb will point in the direction of the surface. In the sketch, the surface is above the plane of the screen. Now at each point of the surface define a unit vector, \( \hat{n} \), directed perpendicular to the surface and directed outward, away from the curve.

We now define the average value of the \( \hat{n} \)-component of the curl to be:

\[
\text{Av}[\hat{n} \cdot \text{cur} \vec{B}] = \frac{1}{\text{Area}} \oint \vec{B} \cdot d\vec{\ell}
\]

where the area is the area of the surface, and we go around the loop keeping it to our left as described above. We then define the curl at a point as:
\[ \left( \text{curl} \ \vec{B} \right) \cdot \hat{n} = \lim_{\text{Area} \to 0} \frac{1}{\text{Area}} \oint \vec{B} \cdot d\vec{r} \]

From our definition of the average value of curl we find:

\[ \oint \vec{B} \cdot d\vec{r} = \text{Area} \left[ \text{Av} \{ \hat{n} \cdot \text{curl} \vec{B} \} \right] \]

\[ = \text{Area} \left[ \frac{1}{\text{Area}} \int_{\text{Area}} (\text{curl} \vec{B}) \cdot \hat{n} dA \right] = \int_{\text{Area}} (\text{curl} \vec{B}) \cdot d\hat{A} \]

This famous result is known as Stokes’ Theorem and will be very useful in what comes later.

Next we see what the curl looks like in Cartesian coordinates. To do so consider a loop lying parallel to the xy plane:

Where the area is \( dx \, dy \)

\[ \oint \vec{B} \cdot d\vec{r} = \int_{1}^{2} + \int_{2}^{3} + \int_{3}^{4} + \int_{4}^{1} = B_x(y)dx + B_y(x + dx)(dy) - B_x(y + dy)dx - B_y(x)dy \]

\[ = -\left[ \frac{B_y(y + dy) - B_x(y)}{dy} \right] dx dy + \left[ \frac{B_y(x + dx) - B_y(x)}{dx} \right] \]

\[ = \left[ \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right] dx dy \]

Since \( \hat{n} = \hat{z} \) in this case we get

\[ (\text{curl} \ \vec{B})_z = \frac{1}{dx \, dy} \int dx \, dy \left( \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) = \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \]

Similarly we find
\[(\text{curl } \vec{B})_x = \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\]

\[(\text{curl } \vec{B})_y = \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\]

Now consider

\[
\vec{V} \times \vec{B} = \left( \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) \times (B_x \hat{x} + B_y \hat{y} + B_z \hat{z})
\]

\[
= \hat{x} \left[ \frac{\partial B_x}{\partial x} \hat{x} + B_x \hat{x}' + \frac{\partial B_y}{\partial y} \hat{y} + B_y \hat{y}' + \frac{\partial B_z}{\partial z} \hat{z} + B_z \hat{z}' \right]
\]

\[
+ \hat{y} \left[ \frac{\partial B_x}{\partial y} \hat{x} + B_x \hat{y}' + \frac{\partial B_y}{\partial y} \hat{y} + B_y \hat{y}' + \frac{\partial B_z}{\partial z} \hat{z} + B_z \hat{z}' \right]
\]

\[
+ \hat{z} \left[ \frac{\partial B_x}{\partial z} \hat{x} + B_x \hat{z}' + \frac{\partial B_y}{\partial z} \hat{y} + B_y \hat{y}' + \frac{\partial B_z}{\partial z} \hat{z} + B_z \hat{z}' \right]
\]

\[
= \hat{z} \frac{\partial B_y}{\partial x} - \hat{y} \frac{\partial B_z}{\partial x} - \hat{z} \frac{\partial B_x}{\partial y} + \hat{x} \frac{\partial B_z}{\partial y} + \hat{y} \frac{\partial B_x}{\partial z} - \hat{x} \frac{\partial B_y}{\partial z}
\]

\[
= \hat{x} \left( \frac{\partial B_z}{\partial y} \right) + \hat{y} \left( \frac{\partial B_x}{\partial z} \right) + \hat{z} \left( \frac{\partial B_y}{\partial x} \right)
\]

\[
= \text{curl } \vec{B}
\]

Hence

\[
\text{curl } \vec{B} = \vec{V} \times \vec{B}
\]

and

\[
\oint \vec{V} \times \vec{B} \cdot d\vec{A} = \oint \vec{B} \cdot d\ell
\]

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