Physics 3740  
Final Exam  
Name: **SOLUTIONS**  
Thursday 31 July 2008  
(Chapters 1 – 7)

You may use your four sheets of notes and formulas, but you must not collaborate with any other person. Do all four problems, showing your method and working clearly (a correct answer alone may not be sufficient). Be sure to include correct units in your answers where appropriate. The number of marks for each part is given in square brackets, [ ], to the right of the question.

1.  

In this problem assume that the rest frame of the earth is the inertial reference frame $S$. A long rocket-ship (whose rest frame is the inertial reference frame $S'$, with the $x'$-axis of $S'$ parallel to the $x$-axis of $S$) moves in the positive $x$-direction relative to $S$ at speed $v$, the origins of the $S$ and $S'$ frames coinciding at time $t = 0 = t'$. A particle of mass $m$ moving in the positive $x'$-direction is measured by observers at rest in the rocket-ship to have (relativistic) momentum $(7/24)m c$. Observers at rest on earth measure the velocity of the same particle in the $x$-direction to be $u_x = (3/5)c$, and they measure the length of the (moving) rocket-ship to be $60$ c-min.

(a) Find the value of $u_{x'}$, the $x'$-component of the velocity of the particle as measured by observers at rest in the rocket frame of reference. Give your answer as a fraction (not a decimal number) times $c$.  

(b) Find the value of $v$, giving your answer as a fraction in lowest terms times $c$ (not as a decimal number).  

(c) Find the $\gamma$ factor for the relative motion of the earth and the rocket-ship, giving your answer as a fraction in lowest terms.  

(d) According to observers on the rocket, what is the length of their ship?  

(e) At the instant when all the clocks in the $S$ frame read $t = T$, the clock at the back end of the rocket-ship reads $t' = 60$ minutes. What is the reading on the clock at the front end of the rocket-ship at that same instant according to observers in $S$?
1. (cont'd) \[ p = \gamma m u = \frac{m u}{\sqrt{1 - \frac{u^2}{c^2}}} \Rightarrow \frac{7}{24} m c = \frac{m u}{\sqrt{1 - \frac{u^2}{c^2}}} \Rightarrow \frac{7}{24} \sqrt{1 - \frac{u^2}{c^2}} = u, \]

so \[ \frac{49}{576} \left( 1 - \frac{u^2}{c^2} \right) = u^2 \Rightarrow \frac{49}{576} - \frac{49 u^2}{576} = u^2 \Rightarrow \]

\[ \frac{49}{576} = u^2 \left( 1 + \frac{49}{576} \right) = u^2 \left( \frac{576 + 49}{576} \right) = u^2 \left( \frac{625}{576} \right), \]

so \[ \frac{u^2}{625} = \frac{49}{576} \Rightarrow u = \frac{7}{25} c; \text{ that is, } u_x' = \frac{7}{25} c. \]

(b) Use the relativistic velocity addition formula \[ u_x' = \frac{u_x - V}{1 - \frac{V u_x}{c^2}}. \]

Since we have been given, \( u_x' = \frac{7}{25} c \) and \( u_x = \frac{3}{5} c \), we have

\[ \frac{7}{25} c = \frac{\frac{3}{5} c - V}{1 - \frac{3}{5} \frac{V}{c^2}} \Rightarrow \frac{7}{25} c \left( 1 - \frac{3}{5} \frac{V}{c} \right) = \frac{3}{5} c - V, \]

or

\[ \frac{7}{25} c - \frac{21}{125} V = \frac{3}{5} c - V \Rightarrow V - \frac{21}{125} V = \frac{3}{5} c - \frac{7}{25} c \Rightarrow V \left( \frac{125}{125} - \frac{21}{125} \right) = c \left( \frac{15}{25} - \frac{7}{25} \right), \]

so \[ V \left( \frac{104}{125} \right) = c \left( \frac{8}{125} \right) \Rightarrow V = \left( \frac{125}{104} \right) \left( \frac{8}{25} \right) c = \left( \frac{5}{104} \right) 8 c = \frac{40}{104} c \Rightarrow V = \frac{5}{13} c. \]

(c) \[ \gamma = \frac{1}{\sqrt{1 - \left( \frac{V}{c} \right)^2}} = \frac{1}{\sqrt{1 - \left( \frac{3}{13} \right)^2}} = \frac{1}{\sqrt{1 - \frac{25}{169}}} = \frac{1}{\sqrt{169 - 25}} = \frac{1}{\sqrt{144}} = \frac{1}{12}, \]

\[ \Rightarrow \gamma = \frac{1}{(\frac{13}{12})}, \text{ so } \gamma = \frac{13}{12}. \]

(d) The ship is at rest with respect to observers on the rocket, so they measure 'proper length' \( L_0 \), whereas the earth-based observers measure the contracted length, \( L \), using the length contraction formula \( L = \frac{L_0}{\gamma} \). We find

\[ L_0 = \gamma L = \frac{13}{12} \times (60 \text{ c-min}) \Rightarrow L_0 = 65 \text{ c-min}. \]
(e) According to observers in $S$, the clocks in $S'$ will not be synchronized, with the 'chasing' clock leading by
\[
\frac{L_0}{c^2} = \frac{(65 \text{ c-min}) \left(\frac{5}{13} c\right)}{c^2} = 25 \text{ min}.
\]
Therefore, the clock at the front of the ship will be reading
\[
60 \text{ min} - 25 \text{ min} = 35 \text{ min}.
\]
2. Suppose that a non-relativistic particle confined to the one-dimensional region $0 < x < L$ has the spatial wave function

$$\psi(x) = \begin{cases} 
C(x - L), & \text{where } C \text{ is a constant, for } 0 < x < L \\
0, & \text{for } x \leq 0 \text{ and } x \geq L 
\end{cases}$$

(a) In terms of $L$, what must be the value of $C$ in order that $\psi(x)$ be normalized? \[11\]

(b) Calculate the expectation value $\langle p^2 \rangle$, that is, the mean value of the square of the linear momentum for this particle. (Recall that $p_{\text{op}} = -i \hbar \frac{d}{dx}$.) \[11\]

\[\text{(a) Normalization Condition: } \int_0^L |\psi|^2 \, dx = 1, \quad \text{or} \]

$$1 = C^2 \int_0^L x^2 (x - L)^2 \, dx = C^2 \int_0^L x^2 \left( x^2 - 2xL + L^2 \right) \, dx$$

$$\Rightarrow 1 = C^2 \left. \int_0^L \left( x^5 - 2x^2L + x^2L^2 \right) \, dx \right|_0^L$$

$$\Rightarrow C^2 \left( \frac{L^5}{5} - \frac{L^5}{2} + \frac{L^5}{3} \right) = 1, \quad \text{or} \quad C^2 \left( \frac{6L^5}{30} - \frac{15L^5}{30} + \frac{10L^5}{30} \right) = 1$$

$$\Rightarrow C^2 \left( \frac{L^5}{30} \right) = 1 \Rightarrow C^2 = \frac{30}{L^5} \Rightarrow C = \pm \sqrt{\frac{30}{L^5}}.$$  

\[\text{(b) } \langle p^2 \rangle = \int_0^L \psi^* p_{\text{op}}^2 \psi \, dx = \int_0^L C(x - L) \left( -\hbar^2 \frac{d^2}{dx^2} \left[ C(x^2 - xL) \right] \right) \, dx \]

$$= -C^2 \left. \int_0^L (x^2 - xL) \hbar^2 \left( \frac{d}{dx} \right)^2 \left[ C(x^2 - xL) \right] \, dx \right|_0^L$$

$$= -2C^2 \left. \int_0^L (x^2 - xL) \hbar^2 \left( \frac{d}{dx} \right)^2 \left[ C(x^2 - xL) \right] \, dx \right|_0^L$$

$$= -2 \left( \frac{30}{L^5} \right) \hbar^2 \left( \frac{x^3}{3} - \frac{x^2L}{2} \right) \bigg|_0^L = -\frac{60\hbar^2}{L^5} \left( \frac{L^3}{3} - \frac{L^3}{2} \right) = -\frac{60\hbar^2}{L^5} \left( \frac{2L^3}{6} - \frac{3L^3}{6} \right),$$

or \[\langle p^2 \rangle = -\frac{60\hbar^2}{L^5} \left( -\frac{L^3}{3} \right) \Rightarrow \langle p^2 \rangle = \frac{10\hbar^2}{L^2} \].
3. The normalized wave function \( \psi(r) \) for a certain hydrogen-like atom is given by

\[
\psi(r) = \frac{1}{\sqrt{2880 \pi a_o^7}} r^2 e^{-r/2a_o}, \text{ where } a_o \text{ is the first Bohr radius.}
\]

(In this problem it may help to use the integral formula \( \int_0^\infty x^n e^{-ax} \, dx = \frac{n!}{a^{n+1}} \).

(a) What is the radial probability density function, \( P(r) \), for this state? (Simplify your answer algebraically as far as possible.)

(b) State in words the physical meaning of \( P(r) \, dr \).

(c) Find the most probable value of \( r \).

(d) Find \( <r> \).

(e) Estimate, correct to two significant figures, the probability that, if an experiment is performed, the electron will be found somewhere between \( r = 2.99 \, a_o \) and \( r = 3.01 \, a_o \). (Use an approximate method. You do not need to perform any actual integration.)

\[
(a) \quad P(r) = 4\pi r^2 |\psi(r)|^2 = 4\pi r^2 \frac{r^4}{(2880\pi a_o^7)} e^{-r/2a_o}
\]

\[
\Rightarrow P(r) = \frac{r^6}{720 \, a_o^7} e^{-r/2a_o}.
\]

(b) \( P(r) \, dr \) is the probability of finding the electron at a distance of between \( r \) and \( r+dr \) from the nucleus.

(c) The most probable value of \( r \) is the one that maximizes the probability density function \( P(r) \). We find it by solving \( \frac{dP(r)}{dr} = 0 \), or

\[
\frac{1}{720 \, a_o^7} \left( 6r^5 e^{-r/2a_o} - \frac{r^6}{a_o^6} e^{-r/2a_o} \right) = 0 \Rightarrow \frac{e^{-r/2a_o}}{720 \, a_o^7} \left( 6 - \frac{r}{a_o} \right) = 0.
\]

This equation has three solutions. Two of them (\( r = 0 \) and \( r = \infty \)) make \( P(r) = 0 \), which is a minimum value. The solution that leads to a maximum of \( P(r) \) is \( 6 - \frac{r}{a_o} = 0 \), or \( r = 6a_o \).
(d) \( \langle r \rangle = \int_0^\infty r P(r) \, dr = \frac{1}{720 a_0^7} \int_0^\infty r^7 e^{-\frac{r}{a_0}} \, dr \)

\[ = \frac{1}{720 a_0^7} \left[ \frac{7!}{(\frac{1}{a_0})^8} \right] \text{using the integral formula given on the previous page} \]

\[ = \frac{5040 a_0^8}{720 a_0^7} \Rightarrow \langle r \rangle = 7a_0 \]

(e) The probability we seek is \( P(r) \, dr \propto P(r) \, \Delta r \)

\[ \propto P(3a_0) (0.02a_0) = \frac{(3a_0)^6 e^{-3a_0/a_0}}{720 a_0^7} \frac{1}{\Delta r} (0.02a_0) \]

\[ = \frac{729 a_0^6 e^{-3} (0.02a_0)}{720 a_0^7} = \frac{14.58 a_0^7 e^{-3}}{720 a_0^7} \]

\[ = \frac{(14.58) (0.049787)}{720} \approx \frac{0.7259}{720} \approx 0.001008, \]

so the probability is about \( 0.0010 \), or \( 0.10 \) per cent

(a) List all possible values of the quantum numbers $\ell$, $j$, and $m_j$ for the single electron in this atom. (Recall that $s = \frac{1}{2}$ for an electron.) I suggest that you do this by filling in the table below: [6]

<table>
<thead>
<tr>
<th>value of $\ell$</th>
<th>possible values of $j$</th>
<th>possible values of $m_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\ell$, $\frac{3}{2}$</td>
<td>$-\frac{1}{2}, \frac{1}{2}$</td>
</tr>
<tr>
<td>1</td>
<td>$\ell$, $\frac{3}{2}$</td>
<td>$-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$</td>
</tr>
<tr>
<td>2</td>
<td>$\ell$, $\frac{3}{2}$</td>
<td>$-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$</td>
</tr>
<tr>
<td>3</td>
<td>$\ell$, $\frac{3}{2}$</td>
<td>$-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$</td>
</tr>
</tbody>
</table>

(b) Given that $\ell$ has its next-to-largest possible value, that $j$ has its maximum possible value (consistent with the value of $\ell$), and that $m_j$ has its minimum possible value (consistent with the value of $j$), calculate:

(i) the magnitude, $J$, of the total angular momentum (in terms of $\hbar$); [4]

(ii) the angle that the total angular momentum vector $\mathbf{J}$ makes with the positive direction of the $z$-axis; [6]

(iii) the expectation value $<J_y^2>$, that is, the mean value $\overline{J_y^2}$. (Hint: Consider the values of $J$ and $J_z$, and use a simple symmetry argument. You do not need to use operators, nor do you need to do any integration.) [6]

(b) $\ell = 2$, $j = \frac{5}{2}$, $m_j = -\frac{5}{2}$

(i) $J = \sqrt{j(j+1)} \frac{\hbar}{\hbar} = \sqrt{\frac{5}{2} \left(\frac{7}{2}\right)} \frac{\hbar}{\hbar} \Rightarrow J = \frac{\sqrt{35}}{2} \frac{\hbar}{\hbar}$

(ii) $\cos \theta = \frac{J_z}{J} = \frac{m_j \hbar}{J} = \frac{-\frac{5}{2} \hbar}{\frac{\sqrt{35}}{2} \hbar} = -\frac{5}{\sqrt{35}} = -\frac{5\sqrt{35}}{35} \approx -0.8452$

$\theta \approx \cos^{-1}(-0.8452) \approx 147.7^\circ$. 

\[\]
(b) \( \vec{J}^2 = J_x^2 + J_y^2 + J_z^2 \)

\[ \Rightarrow J_x^2 + J_y^2 = J^2 - J_z^2 = \left( \frac{\sqrt{3}}{2} \frac{\hbar}{\alpha} \right)^2 - \left( -\frac{\sqrt{3}}{2} \frac{\hbar}{\alpha} \right)^2 \]

\[ \therefore J_x^2 + J_y^2 = \frac{3\sqrt{3}}{4} \frac{\hbar^2}{\alpha^2} - \frac{3\sqrt{3}}{4} \frac{\hbar^2}{\alpha^2} = \frac{1}{4} \frac{\hbar^2}{\alpha^2} = \frac{5}{2} \frac{\hbar^2}{\alpha^2} \]

But symmetry implies that \( \overline{J_x^2} = \overline{J_y^2} \), so we conclude that \( \overline{J_y^2} = \langle J_y^2 \rangle = \frac{5}{4} \frac{\hbar^2}{\alpha^2} \).