You may use your one sheet of notes and formulas, but you must not collaborate with anyone else. Do all four problems, showing your method and working clearly (a correct answer alone is not necessarily sufficient). Be sure to include correct units in your answers where appropriate. The number of marks for each part is given in square brackets, [ ], to the right of the question.

Use the following data for this exam: Boltzmann constant $k = 8.617 \times 10^{-5}$ eV/K; $hc = 1240$ eV·nm; electron rest energy $= mc^2 = 511$ keV; $1 \text{ eV} = 1.6 \times 10^{-19}$ J.

1. (a) A cavity radiator at a temperature of 6000 K has a 10.0 mm diameter hole drilled in its wall. Find the power (in W) radiated through the hole in the range of wavelengths 550 to 551 nm. (Hint: Since the wavelength range is very small, you do not need to perform an actual integration; instead, make a reasonable approximation.) [18]

(b) Calculate the temperature of a cavity radiator having a spectral energy density $u(\lambda)$ at 200 nm that is 3.82 times its spectral energy density at 400 nm. (Hint: Define $x$ to be $\frac{hc}{kT(400 \text{ nm})}$, and then develop an algebraic equation that you can solve for $x$ (or for a simple function of $x$). You may need to use the quadratic formula: $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.) [18]

(a) We know that $R(\lambda)d\lambda = \frac{1}{4}c u(\lambda)d\lambda = \text{power radiated per area in the wavelength range } \lambda \text{ to } \lambda + d\lambda$.

The area of a circular hole of diameter $d$ is $\pi \left(\frac{d}{2}\right)^2 = \frac{1}{4} \pi d^2$.

Since the range of wavelengths is small in this problem, we can approximate the needed integral by a simple product, using the Planck radiation law for $u(\lambda) = \frac{(8\pihc \lambda^{-5})}{(e^{hc/\lambda kT} - 1)}$. Therefore,

\[ P = \text{power radiated thro' the hole in range 550 to 551 nm} = \int_{550}^{551} \text{(area)} R(\lambda)d\lambda \approx (\text{area}) \frac{1}{4}c u(\bar{\lambda}) \Delta \lambda, \]

where $\bar{\lambda} = 550.5 \text{ nm}$ and $\Delta \lambda = (551 - 550) \text{ nm} = 1 \text{ nm}$.

\[ P = \frac{1}{4} \left(\frac{d^2 \pi}{4}\right) \frac{1}{4}c \frac{8\pihc}{\lambda^5} \frac{\Delta \lambda}{(e^{hc/\lambda kT} - 1)} = \frac{(\pi d^2 h c}{2 \lambda^5} \frac{\Delta \lambda}{(e^{hc/\lambda kT} - 1)} \] (P.T.O.)
1. (cont'd)  

(a) Using \( \frac{hc}{k} = 1.240 \text{ eV} \cdot \text{nm} \) and \( k = 8.617 \times 10^{-5} \text{ eV} / K \), this gives

\[
P = \frac{\pi^2 (10^7 \text{ nm})^4 (1240 \text{ eV} \cdot \text{nm}) (1 \text{ nm}) (3 \times 10^{17} \text{ nm})}{2 (550.5 \text{ nm})^5 \left[ e^{\frac{1240 \text{ eV} \cdot \text{nm}}{550.5 \text{ nm}} (8.617 \times 10^{-5} \text{ eV}) (6000 \text{ K})} - 1 \right]} \approx \frac{3.6715 \times 10^{35} (\text{nm})^5}{1.0112 \times 10^{44} (\text{nm})^5} \approx 77
\]

\[\Rightarrow P \approx 4.7 \times 10^{19} \text{ eV/s} \approx 7.5 \text{ W}\]

(b) We require that \( u(200 \text{ nm}) = 3.82 \ u(400 \text{ nm}) \), or

\[
\frac{8\pi hc}{(200 \text{ nm})^5} \left[ \frac{1}{e^{\frac{hc}{kT(200 \text{ nm})}} - 1} \right] = 3.82 \left[ \frac{8\pi hc}{(400 \text{ nm})^5} \left( \frac{1}{e^{\frac{hc}{kT(400 \text{ nm})}} - 1} \right) \right]
\]

If we define \( x := \frac{hc}{kT(400 \text{ nm})} \) then \( e^{\frac{hc}{kT(200 \text{ nm})}} = e^{2x} \), and the equation above simplifies to

\[
\frac{(400 \text{ nm})^5}{(200 \text{ nm})^5} \left( e^x - 1 \right) = 3.82 \left( e^{2x} - 1 \right), \text{ or } 32 (e^x - 1) = 3.82 (e^{2x} - 1)
\]

This last equation is equivalent to \( 3.82 (e^{2x}) - 32e^x + 28.18 = 0 \). Using the quadratic formula to solve this:

\[
e^x = \frac{32 \pm \sqrt{(32)^2 - 4(3.82)(28.18)}}{2(3.82)} = \frac{32 \pm 24.36}{7.64} = \frac{7.64}{7.64} \text{ or } \frac{56.36}{7.64}
\]

Therefore, either \( e^x = 1 \) [which we reject as unphysical, since it implies that \( x = 0 \) and hence \( T \to \infty \)] or \( e^x = \frac{56.36}{7.64} \approx 7.377 \)

\[\Rightarrow x \approx \ln (7.377) \approx 1.998. \text{ From the definition of } x, \text{ we have}
\]

\[
T = \frac{hc}{x \ k(400 \text{ nm})} = \frac{1240 \text{ eV} \cdot \text{nm}}{(1.998)(8.617 \times 10^{-5} \text{ eV/K}) (400 \text{ nm})} \approx 18000 \text{ K}
\]
2. In a photoelectric effect experiment, the stopping potential for electrons emitted from a metal surface illuminated by incident light of wavelength 490 nm is 0.720 V. When the incident wavelength is changed to a new value, the stopping potential is found to be 1.47 V.

(a) What is the work function for the metal surface? [8]

(b) What is the new wavelength of the incident light? [8]

\[
\frac{hc}{\lambda} = eV_0 + \phi \quad \Rightarrow \quad \phi = \frac{hc}{\lambda} - eV_0 = \frac{1240 \text{ eV \cdot nm}}{490 \text{ nm}} - 0.720 \text{ eV},
\]

so \( \phi \approx 2.53 \text{ eV} - 0.72 \text{ eV} \quad \Rightarrow \quad \phi \approx 1.81 \text{ eV} \)

(b) \[
\frac{hc}{\lambda} = eV_0 + \phi = 1.47 \text{ eV} + 1.81 \text{ eV} = 3.28 \text{ eV}, \quad \text{so}
\]

\[
\lambda \approx \frac{hc}{3.28 \text{ eV}} \approx \frac{1240 \text{ eV \cdot nm}}{3.28 \text{ nm}} \quad \Rightarrow \quad \lambda \approx 378 \text{ nm}
\]
3. In the Compton effect, through what angle must a 200 keV photon be scattered by an initially stationary free electron so that the photon will lose 10 per cent of its energy?

For the incident photon, \( \frac{hc}{\lambda} = 200 \text{ keV} \Rightarrow \lambda = \frac{hc}{200 \text{ keV}} \).

The scattered photon has 90 per cent as much energy as the incident photon, or 180 keV. Therefore, for the scattered photon, \( \lambda' = \frac{hc}{180 \text{ keV}} \). By the Compton scattering formula,

\[
\frac{h}{mc} (1 - \cos \theta) = \lambda' - \lambda = hc \left( \frac{1}{180 \text{ keV}} - \frac{1}{200 \text{ keV}} \right)
\]

Multiplying each side of this equation by \( \frac{mc}{h} \) gives

\[
1 - \cos \theta = mc^2 \left( \frac{1}{180 \text{ keV}} - \frac{1}{200 \text{ keV}} \right), \text{ or}
\]

\[
\cos \theta = 1 - \left( \frac{mc^2}{180 \text{ keV}} - \frac{mc^2}{200 \text{ keV}} \right) \approx 1 - \left( \frac{511 \text{ keV}}{180 \text{ keV}} - \frac{511 \text{ keV}}{200 \text{ keV}} \right)
\]

\[
\Rightarrow \cos \theta \approx 0.7161, \text{ so}
\]

\[
\theta = \cos^{-1}(0.7161), \text{ or} \theta \approx 44.3^\circ
\]
4. Use the Bohr model for this problem.

Consider a hypothetical hydrogen-like atom consisting of a nucleus of charge \( Z e \) and an orbiting particle of mass \( m \) and charge \( -e \). Let \( R_\infty \) be the Rydberg constant calculated on the assumption that the nucleus is infinitely massive.

(a) If, in fact, the nucleus has finite mass \( M = 2m \), express the actual value, \( R \), of the Rydberg constant as a fraction times \( R_\infty \). (You do not need to give any numerical value for \( R \) or \( R_\infty \).) \[ 7 \]

(b) If, instead, the nucleus has mass \( M' = 3m \), express the actual value, \( R' \), of the Rydberg constant as a fraction times \( R_\infty \). \[ 4 \]

(c) Assume that the atom described in (a) has atomic number \( Z = N \) and that the atom described in (b) has atomic number \( Z' = N' = 2N \).

Suppose that the atom described in (a) undergoes a transition \( n_i \rightarrow n_f \) and emits a photon of wavelength 630 nm. Calculate the wavelength of the photon that would be omitted by the atom described in (b) as a result of the same transition \( n_i \rightarrow n_f \). \[ 15 \]

\[
\begin{align*}
(a) \quad R_\infty &= \frac{\hbar^2 e^4}{4\pi^2 \hbar^3} \quad \text{and} \quad R &= \frac{\mu k^2 e^4}{4\pi^2 \hbar^3}, \quad \text{where} \quad \mu = \frac{mM}{m+M}, \quad \text{so} \\
R &= \frac{m}{m+M} R_\infty = \left(\frac{M}{m+M}\right) R_\infty = \left(\frac{2m}{m+2m}\right) R_\infty \Rightarrow R = \frac{2}{3} R_\infty.
\end{align*}
\]

(b) Similarly, \( R' = \left(\frac{M'}{m+M'}\right) R_\infty = \left(\frac{3m}{m+3m}\right) R_\infty \Rightarrow R' = \frac{3}{4} R_\infty. \]

(c) \[
\begin{align*}
\frac{1}{\lambda} &= R \frac{Z^2}{n_i^2} \left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) = R N^2 \left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right), \quad \text{and} \\
\frac{1}{\lambda'} &= R' \left(N'\right)^2 \left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) = R' \left(2N\right)^2 \left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right). \quad \text{We therefore have} \\
\frac{\lambda'}{\lambda} &= \left(\frac{1}{\lambda} \right) = \frac{R N^2 \left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)}{R' \left(2N\right)^2 \left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)} = \frac{R N^2}{R' \left(2N\right)^2} = \frac{3}{2} \frac{R_\infty N^2}{R_\infty \left(2N\right)^2} = \frac{3}{4} \frac{R_\infty}{R_\infty} \frac{N^2}{4N^2} = \frac{3}{4} \frac{1}{4} = \frac{3}{4}. \\
\therefore \quad \lambda' &= \frac{2}{3} \lambda = \frac{2}{3} \cdot 630 \text{ nm} \Rightarrow \lambda' = 140 \text{ nm}.
\end{align*}
\]