6-3. In a region of space, a particle has a wave function given by \( \psi(x) = Ae^{-x^2/L^2} \) and energy \( \hbar^2/2mL^2 \), where \( L \) is some length. (a) Find the potential energy as a function of \( x \), and sketch \( V \) versus \( x \). (b) What is the classical potential that has this dependence?

6-3. (a) \[ \frac{d\psi}{dx} = - \left( \frac{x}{L^2} \right) \psi \quad \text{and} \quad \frac{d^2\psi}{dx^2} = \left[ - \frac{x}{L^2} \right] \left[ - \frac{x}{L^2} \right] - \frac{1}{L^2} \] \[ \psi = \frac{x^2}{L^4} \psi - \frac{1}{L^2} \psi \]

Substituting into the time-independent Schrödinger equation,

\[ \left( - \frac{\hbar^2}{2mL^4} + \frac{\hbar^2}{2mL^2} \right) \psi + V(x) = E \psi = \frac{\hbar^2}{2mL^2} \psi \]

Solving for \( V(x) \),
\[ V(x) = \frac{\hbar^2}{2mL^2} - \left( - \frac{\hbar^2 x^2}{2mL^4} + \frac{\hbar^2}{2mL^2} \right) = \frac{\hbar^2 x^2}{2mL^4} = \frac{1}{2} kx^2 \]

where \( k = \hbar^2/mL^4 \). This is the equation of a parabola centered at \( x = 0 \).

(b) The classical system with this dependence is the harmonic oscillator.

6-6. The wave function for a free electron, i.e., one on which no net force acts, is given by \( \psi(x) = A \sin(2.5 \times 10^{10}x) \) where \( x \) is in meters. Compute the electron's (a) momentum, (b) total energy, and (c) de Broglie wavelength.

6-6. (a) For a free electron \( V(x) = 0 \), so

T.I.S. E.: \[ - \frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} = E \psi \quad \text{and} \quad \frac{d^2\psi}{dx^2} = -(2.5 \times 10^{10})^2 \psi \quad \text{for the wave function given.} \]

Substituting into the Schrödinger equation gives:
\[ (2.5 \times 10^{10})^2 (\hbar^2/2m) \psi = E \psi \quad \text{and, since} \quad E = E_k = p^2/2m \quad \text{for a free particle,} \]
\[ p^2 = 2m(2.5 \times 10^{10})^2 (\hbar^2/2m) \quad \text{and} \quad p = (2.5 \times 10^{10}) \hbar = 2.64 \times 10^{-24} \text{kg} \cdot \text{m/s} \]

(b) \[ E = p^2/2m = (2.64 \times 10^{-24} \text{kg} \cdot \text{m/s})^2 / (2)(9.11 \times 10^{-31} \text{kg}) = 3.82 \times 10^{-18} \text{J} \]
\[ = (3.82 \times 10^{-18} \text{J}) (1/1.60 \times 10^{-19} \text{eV}) = 23.9 \text{eV} \]

(c) \[ \lambda = \hbar/p = 6.63 \times 10^{-34} \text{J} \cdot \text{s} / 2.64 \times 10^{-24} \text{kg} \cdot \text{m/s} = 2.5 \times 10^{-10} m = 0.251 \text{nm} \].
6-10. A particle is in the ground state of an infinite square well potential given by Equation 6-21. Find the probability if finding the particle in the interval \( \Delta x = 0.002L \) at (a) \( x = L/2 \), (b) \( x = 2L/3 \), and (c) \( x = L \). (Since \( \Delta x \) is very small, you need not do any integration.)

The ground state wave function is \( (n = 1) \psi_1(x) = \sqrt{2/L} \sin(\pi x / L) \) (Equation 6-32)

The probability of finding the particle in \( \Delta x \) is approximately:

\[
P(x) \Delta x = \frac{2}{L} \sin^2 \left( \frac{\pi x}{L} \right) \Delta x = \frac{2 \Delta x}{L} \sin^2 \left( \frac{\pi x}{L} \right)
\]

(a) for \( x = \frac{L}{2} \) and \( \Delta x = 0.002L \),
\[
P(x) \Delta x = \frac{2(0.002L)}{L} \sin^2 \left( \frac{\pi L}{2L} \right) = 0.004 \sin^2 \left( \frac{\pi}{2} \right) = 0.004.
\]

(b) for \( x = \frac{2L}{3} \),
\[
P(x) \Delta x = \frac{2(0.002L)}{L} \sin^2 \left( \frac{2\pi L}{3L} \right) = 0.004 \sin^2 \left( \frac{2\pi}{3} \right) = 0.004.
\]

(c) for \( x = L \),
\[
P(x) \Delta x = 0.004 \sin^2 \pi = 0.
\]

6-16. The wavelength of light emitted by a ruby laser is 694.3 nm. Assuming that the emission of a photon of this wavelength accompanies the transition of an electron from the \( n = 2 \) level to the \( n = 1 \) level of an infinite square well, compute \( L \) for the well.

\[
E_n = \frac{\hbar^2 n^2}{8mL^2} \quad \text{and} \quad \Delta E_n = E_{n+1} - E_n = \frac{\hbar^2}{8mL^2} (n^2 + 2n + 1 - n^2)
\]

or, \( \Delta E_n = (2n + 1) \frac{\hbar^2}{8mL^2} = \frac{\hbar c}{\lambda} \). In this problem, \( n = 1 \) \( \text{(i.e., } n + 1 = 2 \text{)} \),

so, \( L = \left( \frac{3\lambda h}{8mc} \right)^{1/2} = \left( \frac{3\lambda hc}{8me^2} \right)^{1/2} = \left( \frac{3(694.3 \text{ nm})(1240 \text{ eV} \cdot \text{nm})}{8(0.511 \times 10^6 \text{ eV})} \right)^{1/2} = 0.715 \text{ nm} \).

6-19. An electron moving in a one-dimensional infinite square well is trapped in the \( n = 5 \) state. (a) Show that the probability of finding the electron between \( x = 0.2L \) and \( x = 0.4L \) is 1/5. (b) Compute the probability of finding the electron within the “volume” \( \Delta x = 0.01L \) at \( x = L/2 \).

(a) \( \psi_5(x) = (2/L)^{1/2} \sin(5\pi x / L) \), and \( dx \psi^* \psi \), so
\[
P = \int_{0.2L}^{0.4L} (2/L)^{1/2} \sin^2 (5\pi x / L) \, dx
\]

Letting \( 5\pi x / L = u \), then \( 5\pi \, dx / L = du \) and \( x = 0.2L \Rightarrow u = \pi \) and \( x = 0.4L \Rightarrow u = 2\pi \), so
\[
P = \left( \frac{2}{L} \right) \left( \frac{L}{5\pi} \right)^{2/2} \int_0^{2\pi} \sin^2 u \, du = \left( \frac{2}{L} \right) \left( \frac{L}{5\pi} \right) \left[ \frac{u}{2} - \frac{\sin 2u}{4} \right]_0^{2\pi} = \frac{1}{5} \quad \text{Q.E.D.}
\]
19 (cont'd) (b) The probability of finding the electron in the small interval $\Delta x$ can be estimated without integration:

$$P = \int_{x-\frac{1}{2}\Delta x}^{x+\frac{1}{2}\Delta x} |\psi(x)|^2 \, dx = |\psi^*(x)| \psi(x) \Delta x = \frac{2}{L} \sin^2 \left(\frac{5\pi}{L} x\right) \Delta x.$$  

With $x = \frac{L}{2}$ and $\Delta x = 0.01 L$ this becomes

$$P = \frac{2}{L} \sin^2 \left(\frac{5\pi}{L} \left(\frac{L}{2}\right)\right) (0.01 L) = \frac{2}{L} \sin^2 \left(\frac{5\pi}{4}\right) (0.01 L) = 0.02.$$  

23 6-23. Sketch (a) the wave function and (b) the probability distribution for the $n = 4$ state for the finite square well potential.

(a)

(b)

26 6-26. Using arguments concerning curvature, wavelength, and amplitude, sketch very carefully the wave function corresponding to a particle with energy $E$ in the finite potential well shown in Figure 6-32.

For $V_2 > E > V_1$: $x_1$ is where $V = 0 \rightarrow V_1$  
and $x_2$ is where $V_1 \rightarrow V_2$.

From $-\infty$ to 0 and $x_2$ to $+\infty$: $\psi$ is exponential.

0 to $x_1$: $\psi$ is oscillatory; $E_k$ is large so $p$ is large and $\lambda$ is small; amplitude is small because $E_k$ hence $v$, is large.

$x_1$ to $x_2$: $\psi$ is oscillatory; $E_k$ is small so $p$ is small and $\lambda$ is large; amplitude is large because $E_k$, hence $v$, is small.
6.33. Compute $\langle x \rangle$ and $\langle x^2 \rangle$ for the ground state of a harmonic oscillator (Equation 6.58). Use $A_0 = (m\omega/h\pi)^{1/4}$.

$$\psi_0(x) = A_0 e^{-mwx^2/2h}$$

where $A_0 = (m\omega/h\pi)^{1/4}$

$$\langle x \rangle = \int A_0^2 x e^{-mwx^2/2h} dx.$$ Letting $u^2 = m\omega x^2/h$, $2udu = (m\omega/h)(2xdx)$, and thus, $(m\omega/h)^{-1}udu = xdx$; limits are unchanged.

$$\therefore \langle x \rangle = A_0^2 (h/m\omega) \int u e^{-u^2} du = 0$$ (Note that the symmetry of $V(x)$ would also tell us that $\langle x \rangle = 0$.)

$$\langle x^2 \rangle = \int A_0^2 x^2 e^{-mwx^2/h} dx$$

$$= A_0^2 (h/m\omega)^{3/2} \int u^2 e^{-u^2} du = 2A_0^2 (h/m\omega)^{3/2} \int_0^{\infty} u^2 e^{-u^2} du$$

$$= 2A_0^2 (h/m\omega)^{3/2} \sqrt{\pi}/4 = (m\omega/h\pi)^{1/2} (h/m\omega)^{3/2} \sqrt{\pi}/2 = h/(2m\omega)$$

6.36. For the harmonic oscillator ground state $n = 0$ the Hermite polynomial $H_n(x)$ in Equation 6.57 is given by $H_0(x) = 1$. Find (a) the normalization constant $C_0$, (b) $\langle x^2 \rangle$, and (c) $\langle V(x) \rangle$ for this state. (Hint: Use Table B1-1 to compute the needed integrals.)

Putting $H_0(x) = 1$ in equation (6.57) gives for the ground state $\psi_0(x)$ of the simple harmonic oscillator:

$$\psi_0(x) = C_0 e^{-mwx^2/2h}$$

(a) The normalisation condition, equation (6.20), then gives

$$1 = \int_{-\infty}^{\infty} |\psi_0(x)|^2 dx = \int_{-\infty}^{\infty} |C_0|^2 e^{-mwx^2/2h} dx = |C_0|^2 2 \int_0^{\infty} e^{-\frac{mwx^2}{2h}} dx$$

$$= |C_0|^2 2 I_o, \quad \text{from Table B1-1, p. AP16, with } \lambda = m\omega/h.$$ 

$$= |C_0|^2 2 \left[ \frac{1}{2} \pi^{1/2} \frac{\sqrt{5}}{m\omega} \right] = |C_0|^2 \sqrt{\frac{\pi}{m\omega}} = 1 \Rightarrow |C_0|^2 = \frac{(m\omega)^{1/4}}{(\pi^{1/2} h)}.$$

Notice that we had to integrate from $-\infty$ to $\infty$, rather than just $-A$ to $A$. This is because the harmonic oscillator wave functions (6.53) do not abruptly
(b) \( \langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 |\psi_0(x)|^2 \, dx = \int_{-\infty}^{\infty} x^2 e^{-\frac{m \omega x^2}{\hbar}} \, dx \)

\[ = \left( \frac{m \omega}{\hbar} \right)^{\frac{1}{2}} \int_{-\infty}^{\infty} x^2 e^{-\frac{m \omega x^2}{\hbar}} \, dx = \left( \frac{m \omega}{\hbar} \right)^{\frac{1}{2}} \frac{2 \pi}{\omega} , \text{ with } \lambda = \frac{m \omega}{\hbar}. \]

\[ \therefore \langle x^2 \rangle = \left( \frac{m \omega}{\hbar} \right)^{\frac{1}{2}} \frac{2 \pi}{\sqrt{m \omega}} \]

\[ \Rightarrow \langle x^2 \rangle = \frac{1}{2} \ \frac{\hbar}{m \omega} \]

(c) \[ \langle N(x) \rangle = \langle \frac{1}{2} m \omega x^2 \rangle = \frac{1}{2} m \omega \langle x^2 \rangle = \frac{1}{2} m \omega \left( \frac{1}{2} \ \frac{\hbar}{m \omega} \right) \]

\[ = \frac{1}{4} \ \frac{\hbar \omega}{m} \]

\[ \therefore \langle N(x) \rangle = \frac{1}{4} \ \frac{\hbar \omega}{m} = \frac{1}{2} \cdot \left( \frac{\hbar}{2m} \right) = \frac{1}{2} E_0 = \frac{1}{2} \text{ of the total energy of the system, in agreement with the classical result.} \]

\[ \text{by eqn 6.56} \]

6.39 Compute the spacing between adjacent energy levels per unit energy, i.e., \( \Delta E/E_n \) for the quantum harmonic oscillator and show that the result agrees with Bohr’s correspondence principle (see Section 4.3) by letting \( n \to \infty \).

\[ E_n = (n+1/2) \hbar \omega \]

\[ E_{n+1} = (n+3/2) \hbar \omega \]

\[ E_{n+1} - E_n = \Delta E_n = (n+3/2-n-1/2) \hbar \omega = \hbar \omega = \Delta E_n \], so

\[ \Delta E_n/E_n = \frac{\hbar \omega}{\hbar \omega (n+1/2)} = 1/(n+1/2). \]

\[ \therefore \lim_{n \to \infty} \frac{\Delta E_n}{E_n} = \lim_{n \to \infty} \left( \frac{1}{n+1/2} \right) = 0 \text{ In agreement with the correspondence principle.} \]

6.44 Suppose that the potential jumps from zero to \(-V_0\) at \(x = 0\) so that the free particle speeds up instead of slowing down. The wave number for the incident particle is again \( k_1 \), and the total energy is \(2V_0\). (a) What is the wave number for the particle in the region of positive \(x\)? (b) Calculate the reflection coefficient \( R \) at the potential step. (c) What is the transmission coefficient \( T \)? (d) If one million particles with wave number \( k_1 \) are incident upon the potential step, how many particles are expected to continue along in the positive \(x\) direction? How does this compare with the classical prediction?
(44) (Cont'd) Using results from pages 251 and 253:

(a) For \( x > 0 \), \( \hbar^2 k_z^2 / 2m - V_0 = E = \hbar^2 k_1^2 / 2m = 2V_0 \)

So, \( k_2 = (6m V_0)^{1/2} / \hbar \). Since \( k_1 = (4m V_0)^{1/2} / \hbar \), then \( k_2 = \sqrt{3/2} k_1 \).

(b) \( R = \frac{k_1^2 - k_2^2}{k_1^2 + k_2^2} = \frac{1 - \sqrt{3/2}}{1 + \sqrt{3/2}} \approx 0.0102 \). Or 1.02% are reflected at \( x = 0 \).

(c) \( T = 1 - R = 0.99 \)

(d) 99% of the particles, or \( 0.99 \times 10^6 = 9.9 \times 10^5 \) continue in the \(+x\) direction. Classically, 100% would continue on.

6-48 A beam of electrons, each with kinetic energy \( E = 2.0 \text{ eV} \), is incident on a potential barrier with \( V_0 = 6.5 \text{ eV} \) and width \( 5.0 \times 10^{-10} \text{ m} \). (See Figure 6-27.) What fraction of the electrons in the beam will be transmitted through the barrier?

Using Equation 6-76, \( T = \frac{16E}{V_0} \left( 1 - \frac{E}{V_0} \right) e^{-2\alpha} \) where \( E = 2.0 \text{ eV}, V_0 = 6.5 \text{ eV}, \)

\[
\alpha = \sqrt{\frac{2m(V_0 - E)}{\hbar}} = \sqrt{\frac{2mc^2(V_0 - E)}{\hbar c}} = \frac{\sqrt{2(511.000 \text{ eV})(4.5 \text{ eV})}}{197.3 \text{ eV} \cdot \text{nm}} \approx 10.87 \text{ nm}^{-1}
\]

and \( \alpha = 0.50 \text{ nm} \):

\[
T = 16 \left( \frac{2.0}{6.5} \right) \left( 1 - \frac{2.0}{6.5} \right) e^{-2(10.87)(0.5)} \approx 6.5 \times 10^{-3} \text{.}
\]

(Equation 6-75 yields \( T = 6.6 \times 10^{-2} \).)