METHOD OF IMAGES

The method is based on the fact that any solution of Poisson’s Equation that satisfies the appropriate boundary conditions is the unique solution. There are three possible boundary conditions that assure this result. If the voltage is known on a closed surface (Dirichlet conditions) bounding the volume in question, the solution is unique. If the normal component of the electric field is known everywhere on the same closed surface the solution is unique except for an arbitrary constant. This is Neumann Conditions. If one or the other of these is known everywhere on the closed surface the solution is unique.

Case 1: Grounded Infinite Plane

Consider a conducting plane occupying the region \( x \leq 0 \).

We make the voltage 0 on the plane and on the half sphere at \( \infty \). This is called “grounding.” Physically “grounding” means connecting the plane to a very large source of charge so that whatever charge the conductor gives or receives from the source does not affect the course—like pouring a cup of water into the ocean.

We now place a point charge \( q \) a distance \( x \) in front of the plane and ask what the force on the charge will be.

We need a solution of Poisson’s equation which makes \( V = 0 \) everywhere on the plane. We can obviously accomplish this by imagining placing a charge \(-q\) a distance \( x \) behind the surface as shown. This has not changed the charge distribution in region (1) and does make \( V = 0 \) everywhere on the surface of the plane. Since the half sphere at \( \infty \) is infinitely far away it keeps the voltage there at 0. Then the force on \( q \) is
\[ \vec{F} = -\frac{kq^2}{4x^2} \hat{x} \]

We can find the charge on the surface of the plane from Gauss’s Law

\[ \oint \vec{E} \cdot d\vec{A} = E_\perp A = 4\pi k \sigma A \rightarrow \sigma = \frac{E_\perp}{4\pi k} \]

But

\[ E_\perp = -\frac{2kqx}{(\rho^2 + x^2)^{3/2}} \]

\[ \therefore \sigma = -\frac{2kqx}{4\pi k(\rho^2 + x^2)^{3/2}} = -\frac{-qx}{2\pi(\rho^2 + x^2)^{3/2}} \]

Then the total charge on the plate is

\[ Q = \int_0^\infty \sigma 2\pi \rho d\rho = -q \int_0^\infty \frac{\rho d\rho}{(\rho^2 + x^2)^{3/2}} \bigg|_0^\infty = -q \]  

You should ask yourself why this is the expected answer (why not \( q/2 \) with half the field lines from \( q \) going to the half sphere of infinity). To answer this remember our explanation for the force on the surface of a conductor.

Case 2: Isolated Conducting Plane

Now the voltage will still have to be constant on the surface of the conductor. We arrange this as before with the same image charge. But now we cannot have a net charge on the plane. We arrange this by placing a uniform charge density
\[ \sigma_i = \frac{q}{\pi R^2} \quad R = \text{radius of plate} \]

Then

\[
\vec{F} = -\frac{kq^2}{4x^2} \left[ \frac{\kappa q}{\pi R^2} \int_0^R \frac{x}{\rho^2 + x^2}^{3/2} \, d\rho \right]\left(\rho^2 + x^2\right)^{1/2} - \frac{1}{4x^2} \int_0^R \frac{2kqx}{R^2} \frac{\rho^2 + x^2}{\left(\rho^2 + x^2\right)^{1/2}} \, d\rho \left|_0^R \right.
\]

\[
= -\frac{kq^2}{4x^2} + \frac{2kqx}{R^2} \left( \frac{1}{2} - \frac{1}{\left[\rho^2 + x^2\right]^{1/2}} \right) \xrightarrow{R \rightarrow \infty} -\frac{kq^2}{4x^2}
\]

Physically what happens is that the surface charge is now drawn from the other side of the plate, leaving a net + charge behind. But this charge can’t effect the field in front of the plane—remember the shielding properties of conductors.

Case 3: Spheres

Next consider a grounded conducting sphere with a point charge \( q \) outside the sphere. We need to make the sphere—and a surrounding sphere at \( \infty \)—have \( V = 0 \). We do this by placing an image charge, \(-q'\), at a distance, \( a \), from the center of the sphere—as shown

Realizing that a plane is a sphere in the limit \( R \rightarrow \infty \), we expect

\[ a = \alpha r \quad q' = \beta q \]

where both \( \alpha \) and \( \beta \) are dimensionless constants. For \( \beta \) we note that the only dimensionless constant available is \( R/r \). Hence we guess
\[ q' = \frac{R}{r} q \]

Then we need to have \( V = 0 \) at the point where the line from \( q \) to the center of the sphere meets the sphere

\[
\frac{kq}{r - R} = \frac{kq}{R - a} \Rightarrow R - a = \frac{R}{r} (r - R)
\]

\[ \therefore a = \frac{R^2}{r} \]

Let’s try it. At angle \( \theta \) we have

\[
V = \frac{kq}{\left( r^2 + R^2 - 2rR \cos \theta \right)^{1/2}} - \frac{kq}{r} \frac{R}{r} \left[ \frac{R^4}{r^2} - 2 \frac{R^3}{r} \cos \theta \right]^{1/2}
\]

\[
= \frac{kq}{\left( r^2 + R^2 - 2r \cos \theta \right)^{1/2}} - \frac{kq}{r} \frac{R}{r} \left[ r^2 + R^2 - 2rR \cos \theta \right]^{1/2} \frac{R}{r} = 0
\]

So our guess is correct and an image charge, \(-q\ R/r\), located at \( a = R^2/r \) from the center along the line from the center of the sphere to \( q \) will make the sphere surface have \( V = 0 \). As before we can find the surface charge density from

\[ E_\perp = 4\pi k\sigma \]

Hence we need \( E_\perp \) due to the two charges
\[ E_\perp = \frac{kq}{R} \left[ \hat{x}(R \cos \theta - a) + \hat{y}R \sin \theta \right] \cdot \left( \hat{x} \cos \theta + \hat{y} \sin \theta \right) \cdot \left( R^2 + a^2 - 2aR \cos \theta \right)^{3/2} \]

\[ + \frac{kq}{R} \left[ \hat{x}(R \cos \theta - r) + \hat{y}R \sin \theta \right] \cdot \left( \hat{x} \cos \theta + \hat{y} \sin \theta \right) \cdot \left( R^2 + r^2 - 2rR \cos \theta \right)^{3/2} \]

\[ = -\frac{kq}{r} \frac{R}{r} \left( R - a \cos \theta \right) \left( R^2 + \frac{R^4}{r^2} - 2\frac{R^3}{r} \cos \theta \right)^{3/2} + \frac{kq}{r} \left( R - r \cos \theta \right) \left( R^2 + r^2 - 2rR \cos \theta \right)^{3/2} \]

\[ = -\frac{kq}{R} \frac{r^2}{R} \left( R - a \cos \theta \right) - kq \left( r \cos \theta - R \right) \left( r^2 + R^2 - 2rR \cos \theta \right)^{3/2} \]

\[ = -\frac{kq}{R} \frac{r^2}{R} \left( r \cos \theta + R \cos \theta - R \right) \left( r^2 + R^2 - 2rR \cos \theta \right)^{3/2} = -\frac{kq}{R} \left( \frac{r^2}{R} - R \right) \left( r^2 + R^2 - 2rR \cos \theta \right)^{3/2} \]

\[ \sigma = -\frac{q}{4\pi} \frac{\left( \frac{r^2}{R} - R \right)}{\left( r^2 + R^2 - 2rR \cos \theta \right)^{3/2}} \]

The total charge on the sphere is
\[ Q = \int_0^\pi R^2 \sin \theta 2 \pi \sigma = -\frac{qR^2}{2} \left( \frac{r^2}{R} - R \right) \int_0^\pi \frac{\sin \theta d\theta}{\left( r^2 + R^2 - 2rR \cos \theta \right)^{3/2}} \]

\[ = \frac{qR^2}{2} \left( \frac{r^2}{R} - R \right) \left. \frac{1}{rR \left( r^2 + R^2 - 2rR \cos \theta \right)^{1/2}} \right|_0^\pi = -\frac{qR^2}{2} \left( \frac{r^2}{R} - R \right) \frac{1}{rR} \left[ \frac{1}{r - R} - \frac{1}{r + R} \right] \]

\[ = -\frac{qR}{2r} \left( \frac{r^2}{R} - R \right) \frac{2R}{r^2 - R^2} = -\frac{qR}{r} \]

as expected. As a further check we note that in the limit \( R \to \infty \) this should reduce to the result for the plane

\[ Q = -q \frac{R}{R + x} \xrightarrow{R \to \infty} -q \]

as before.

The force on the charge \( q \) is, of course,

\[ F = -\frac{kqq'}{(r - a)^2} = -\frac{kq^2 R}{r} = -\frac{kq^2 R}{r^3 \left( 1 - \frac{R^2}{r^2} \right)^2} \]

Hence it is attractive. Checking the \( R \to \infty \) limit we have

\[ F = -\frac{kq^2 R}{(R + x)^3 \left( 1 - \frac{R^2}{(R + x)^2} \right)^2} = \frac{-kq^2 R}{R^3 \left( 1 + \frac{x}{R} \right)^3} \left[ \frac{1}{1 - \frac{1}{\left( 1 + \frac{x^2}{R^2} \right)}} \right] \]

\[ = -\frac{kq^2}{R^2} \left( 1 - 3 \frac{x}{R} \right) \frac{1}{\left[ 1 - 1 + 2 \frac{x}{R} \right]^2} = -\frac{kq^2}{R^2} \left( 1 - 3 \frac{x}{R} \right) \frac{R^2}{4x^2} \xrightarrow{R \to \infty} \frac{q^2 k}{4x^2} \]
as expected.

Case 4: Isolated Sphere

An isolated uncharged sphere must remain so. Thus we need to add a charge

$$-Q = q \frac{R}{r}$$

at the center of the sphere to reduce the charge on the surface to 0. Thus.

with the resulting force

$$F = k q^2 \frac{R}{r} \left[ \frac{1}{r^2} - \frac{1}{\left( \frac{R^2}{r} \right)^2} \right]$$

We can also check these in the $R \rightarrow \infty$ limit

$$\sigma = -q \frac{\left( \frac{r^2}{R} - \frac{R}{r} \right)}{4\pi \left( r^2 + R^2 - 2rR \cos \theta \right)^{3/2}} + \frac{q}{4\pi R r}$$
\[
F \rightarrow \frac{kq^2R}{R + x} \left[ \frac{1}{(R + x)^2} - \frac{(R + x)^2}{\{(R + x)^2 - R^2\}^2} \right] = \frac{kq^2}{R^2} \left[ \frac{1}{1 + \frac{x}{R}} \right] \left[ \frac{1}{\{1 - 2 \frac{x}{R}\}} - \frac{R^2 \left(1 + 2 \frac{x}{R}\right)}{\{2 + R + x^2\}^2} \right]
\]

\[
= kq^2 \left(1 - \frac{x}{R}\right) \frac{1}{R^2} - \frac{2x}{R^3} \left(1 + \frac{2x}{R}\right) \frac{1}{\left(2x + \frac{x^2}{R}\right)^2} \xrightarrow{r \rightarrow \infty} \frac{kq^2}{4x^2}
\]
as expected.