Example

Now suppose \( \hat{A} = x \) and \( \hat{B} = \frac{\hbar}{i} \frac{d}{dx} \) (operators for momentum).

Then 
\[
\left[ \hat{A}, \hat{B} \right] = \left[ x, \frac{\hbar}{i} \frac{d}{dx} \right] = \frac{\hbar}{i} \frac{d}{dx} x - x \frac{\hbar}{i} \frac{d}{dx} = \frac{\hbar}{i} \frac{d}{dx} - \frac{\hbar}{i} x \frac{d}{dx} = \frac{\hbar}{i} \frac{d}{dx} - \frac{\hbar}{i} \frac{d}{dx} = \frac{\hbar}{i} \frac{d}{dx}
\]

Then 
\[
\sigma_x \sigma_y \geq \left( \frac{1}{2\hbar} \right)^2 = \left( \frac{\hbar}{2} \right)^2 = \hbar \text{ Planck's constant.}
\]

From the generalized uncertainty principle, we see that there will be an uncertainty principle for every pair of operators that do not commute. These are called incompatible operators.

Non-commuting operators cannot have a complete set of common eigenvalues.

(For proof, see Problem 3.15)

This generalized uncertainty principle can also be used for other incompatible operators (e.g., angular momentum and energy) that assumption did we make in deriving the uncertainty principle. Many assumptions were that the operators are Hamiltonian (corresponding to observables) and we assumed the operators to commute.

Minimum Uncertainty Wave Packet - What is the shape of a wave packet that will give minimum uncertainty?

In our derivation, we used inequality: 1) Schwarz's inequality \( \langle f|H|g \rangle \geq \frac{1}{2} \langle f|f \rangle \langle g|g \rangle \),

Schwarz Inequality because inequality \( \langle f|g \rangle \leq \frac{1}{2} \langle f|f \rangle \langle g|g \rangle \).

2) Because \( \text{Schwarz inequality}\), \( \langle f|g \rangle = 0 \) implies \( \langle f| f \rangle = 0 \).

For \( f(\xi) \) and \( g(\xi) \).

Schwarz's inequality: \( \langle f|g \rangle \leq \sqrt{\langle f|f \rangle \langle g|g \rangle} \).

Let \( \langle f|g \rangle = 0 \), where \( f \) \& \( g \) are not zero and unique.

\[ \langle f|g \rangle = \int \overline{f}(\xi) g(\xi) d\xi \]

\[ \langle f|g \rangle = 0 \]
Solution to this differential eqn. is a Gaussian function multiplied by a Gaussian:

\[ \psi(x) = A e^{-\frac{(x-x_0)^2}{2\sigma^2}} \]

So, uncertainty wave packet in a Gaussian (i.e., ground state of harmonic oscillator)

Energy-time uncertainty principle

Often stated as \( \Delta E \Delta t \geq \hbar \), but what is the meaning of this? Particularly, in contrast to position, momentum, energy, all of which are properties of a system and can be measured, it is not a property of the system, but it is an independent variable. What our measurements is that \( \Delta t \) is not a standard deviation of a set of time measurements; rather, it is the time required for the system to change substantially.

Consider an observable \( \mathcal{O}(x,p,t) \), calculate \( \frac{\partial}{\partial t} \langle \mathcal{O} \rangle \):

\[
\frac{d}{dt} \langle \mathcal{O} \rangle = \frac{1}{\hbar} \left[ \langle \mathcal{O} \rangle, \mathcal{H} \right]
\]

Schrödinger Eqn. \( \frac{\partial \psi}{\partial t} = -i \frac{\mathcal{H}}{\hbar} \psi \), \( \mathcal{H} = \frac{p^2}{2m} + V \) is Hamiltonian.

So \( \frac{\partial \psi}{\partial t} = \frac{\partial \psi}{\partial t} \psi \Rightarrow \frac{d}{dt} \langle \mathcal{O} \rangle = -i \left( \langle \mathcal{O} \rangle \frac{\partial}{\partial \psi} + \frac{\partial}{\partial \psi} \langle \mathcal{O} \rangle \right) \]

But \( \mathcal{H} = \text{Hermitian} \) \( \Rightarrow \langle \mathcal{O} \mathcal{O} \rangle = \langle \mathcal{O} \rangle \langle \mathcal{O} \rangle \]

Thus \( \frac{d}{dt} \langle \mathcal{O} \rangle = \frac{1}{\hbar} \left[ \langle \mathcal{O} \rangle, \mathcal{H} \right] + \frac{\partial}{\partial t} \langle \mathcal{O} \rangle \)
Most operators in physics do not have an explicit time dependence. For them, \( \frac{\partial \mathcal{Q}}{\partial t} = 0 \). In this case, if \( \mathcal{Q} \) also commutes with the Hamiltonian operator \( \hat{H} \), then the corresponding observable \( \mathcal{Q} \) is a constant of the motion and \( \frac{\partial}{\partial t} \langle \mathcal{Q} \rangle = 0 \).

Look at the generalized form: \( T_A T_B \geq \left( \frac{\langle [A, B] \rangle}{2} \right)^2 \)

Valid for any Hermitian operators \( A \) and \( B \).

Pick \( A = \hat{H} \) and \( B = \mathcal{Q} \) and do not have explicit time dependence. We then have \( T_\hbar T_\hbar \geq \left( \frac{T_\hbar}{2} \right)^2 \).

Let \( \Delta E = T_\hbar \) = standard deviation of set of measurements at energy

Define \( \Delta t = \frac{T_\hbar}{\alpha} \) or \( \Delta t = \frac{\Delta \langle \mathcal{Q} \rangle}{\alpha} \).

Writing \( \Delta E \Delta t \geq \frac{\hbar}{2} \), note \( \Delta t \) = time required for \( \langle \mathcal{Q} \rangle \) to change by one standard deviation.

\( \Delta t \) may be quite different for different observables. However, if one observable changes very rapidly, the uncertainty in energy \( \Delta E \) is large.

Various examples in your text. Read Example 3.5, 3.6, 3.7.
Dirac Notation

As we have seen, the wave function can be expressed as a linear combination of functions that form a complete set. There are many possible sets of such functions (e.g., harmonic oscillators, spherical harmonics, etc.) Each of these is called a basis and these bases are akin to coordinate systems in real space.

Think of the wave function as a vector, designated by $|\psi\rangle$. A basis vector (called a ket), which can be expressed as a linear combination or "attitude" of the various components of the basis (denoted by $e_n$). Then $|\psi\rangle$ can be expressed in terms of $e_n$ as

$$|\psi\rangle = \sum_n a_n |e_n\rangle,$$

with $a_n = \langle\psi|e_n\rangle$ (components of vector).

Now consider an operator $O$ acting on $|\psi\rangle$. It can be viewed as transforming the vector $|\psi\rangle$ to a new vector $|\phi\rangle$.

$$|\phi\rangle = O |\psi\rangle = \sum_n a_n |e_n\rangle = \sum_n b_n |e_n\rangle$$

Take inner product with $e_m$ and get

$$\langle\phi|e_m\rangle = \sum_n a_n \langle\psi|e_n|e_m\rangle = \sum_n b_n \langle e_n|e_m\rangle = \sum_n a_n Q_{mn} a_n$$

So we can think of the components $Q_{mn}$ as components of an $N \times N$ matrix. Then we can use the rules of matrix multiplication to go from one vector to another.

Thus, for every ket vector in a particular Hilbert space, we can define a new vector $\phi$ in a dual-space spanned by ket vectors $e_n$. This vector $\langle\phi|$ is called a bra. Now, the dual vector to the ket vector is defined by $\langle\phi|e_n\rangle$. One can operate "with" a bra vector from the left of a ket vector, or the other way around, but when a bra hits a ket vector from the left, it always a scalar (number).
Now \( |d> = \sum_{n} a_n |e_n> \) with \( a_n = \langle e_n | d> \)

\[
\sum_{n} \langle e_n | d> \langle e_n | d> = \sum_{n} |e_n><e_n| \Rightarrow \sum_{n} |e_n><e_n| = I
\]

Since \( \langle e_n | d> \) is just a number it can be written to go left or right of \( |e_n><e_n| \). It makes no difference.

We can rewrite this as saying \( \sum_{n} |e_n><e_n| = I \)

This is called closure and is only valid if \( e_n \) states \( |e_n> \) from a complete set. As we will see later, this closure relation can be very useful.

Now consider an operator \( \hat{P} \) defined by

\[
\hat{P} = |d><d|
\]

\( |d><d| \) is a linear operator, \( \langle d| \) represents the fraction of \( |d> \) that's along \( |d> \) or, if you prefer, the fraction of \( |d> \) that's along \( |d> \).

So \( \hat{P} |d> \) represents the "amount" of \( |d> \) that's in the \( |d> \) direction in Hilbert space.

Matrix Mechanics

We can view \( \hat{H} \) as an eigenvalue problem, if we express the wave function in terms of a discrete basis \( \{ |e_n> \} \).

Express the wave function in terms of a discrete basis \( \{ |e_n> \} \).

And express the Hamiltonian as a matrix, then determining the eigenvalues.

\[
\begin{bmatrix}
H_{1,1} & H_{1,2} & \cdots & H_{1,N} \\
H_{2,1} & H_{2,2} & \cdots & H_{2,N} \\
\vdots & \vdots & \ddots & \vdots \\
H_{N,1} & H_{N,2} & \cdots & H_{N,N}
\end{bmatrix}
\]

Once the eigenvalues are known, it is easy to obtain the superposition. This is the approach.