Examination booklets have been provided for recording your work and your solutions. Please note that there is a separate booklet for each numbered question (i.e., use booklet #1 for problem #1, etc.).

To receive full credit, not only should the correct solutions be given, but a sufficient number of steps should be given so that a faculty grader can follow your reasoning. Define all algebraic symbols that you introduce. If you are short of time it may be helpful to give a clear outline of the steps you intended to complete to reach a solution. In some of the questions with multiple parts you will need the answer to an earlier part in order to work a later part. If you fail to solve the earlier part you may represent its answer with an algebraic symbol and proceed to give an algebraic answer to the later part. This is a closed book exam: No notes, books, or other records should be consulted. YOU MAY ONLY USE THE CALCULATORS PROVIDED. The total of 250 points is divided equally among the ten questions of the examination.

All work done on scratch paper should be NEATLY transferred to answer booklets.

SESSION 1
Problem 1 - Mechanics

A massless string is wound around a solid spool of uniform density, radius R and mass M. Under the influence of gravity (g), it is released from rest when the string is tight and the top of the string is attached to a fixed bar. When the spool is released, there is a small dot (massless) found exactly at the top of the spool as seen in the figure. In this problem, you may neglect any effect of friction. Put your answers in terms of g, M, R and displacement d.

(a) [10 pts.] Calculate the tension in the string in terms of g, M and R.

(b) [10 pts.] What is the speed of the center of mass of the spool as a function of height? (Solve without using forces.)

(c) [5 pts.] What is the total velocity vector \( \vec{V} \) (in \( \hat{x} \) and \( \hat{y} \) coordinates) of the dot when the dot is first found at the lowest point on the spool (as shown in the figure)?
**Problem 1**

(a) 

(1) \[ F_t = Mg - T = Ma_{cm} \]

(2) \[ \omega = RT = \frac{I}{m} \frac{dw}{dt} \]

\[ V_{cm} = R \omega \]

\[ a_{cm} = \frac{R \omega}{dt} \]

\[ TR = \frac{I a_{cm}}{R} \]

\[ \frac{TR^2}{I} = a_{cm} = \frac{Mg - \frac{T}{I}}{M} = g - \frac{T}{M} \]

\[ T \left( \frac{R^2}{I} + \frac{1}{M} \right) = g \]

\[ T = \frac{g}{\left( \frac{R^2}{I} + \frac{1}{M} \right)} \]

Rotational Inertia

\[ I = \frac{1}{2} MR^2 \]

\[ T = \frac{\frac{g}{\left( \frac{2}{M} + \frac{1}{M} \right)}}{3} = \frac{Mg}{3} \]
b) \[ MgL = \frac{1}{2}M V_{cm}^2 + \frac{1}{2} I \omega^2 \]

\[ V_{cm} = R \omega \quad \omega = \frac{V_{cm}}{R} \]

\[ MgL = \frac{1}{2} M V_{cm}^2 + \frac{1}{2} \left( \frac{1}{2} MR^2 \right) \left( \frac{V_{cm}}{R} \right)^2 \]

\[ MgL = \frac{3}{4} M V_{cm}^2 \]

\[ V_{cm} = \sqrt{\frac{4}{3} gL} \]
Problem 1
As spool rotates by 1/2 rotation the displacement changes by
\[ d = \pi R \]

From Part B
\[ V_{cm} = \sqrt{\frac{4}{3} g \cdot d} \]
\[ V_{cm} = \sqrt{\frac{4}{3} g \cdot \pi R} \quad \text{Vertical Velocity} \]

TANGENTIAL VELOCITY (ABOUT CENTER OF MASS)

By inspection - the magnitude of the TANGENTIAL VELOCITY is equal to the magnitude of the VELLOCITY OF THE CENTER OF MASS

\[ \left| V_{\text{tang}} \right| = \left| V_{cm} \right| \]
\[ \vec{V}_{\text{total}} = \vec{V}_{cm} + \vec{V}_{\text{tang}} \]

Total velocity at \( d = \pi R \)
\[ \vec{V} = -\sqrt{\frac{4}{3} g \pi R} \hat{x} + \sqrt{\frac{4}{3} g \pi R} \hat{y} \]
Problem 2 - Quantum Mechanics

A particle of mass $m$ is in an infinite square potential well of width $2a$:

$$V(x) = \begin{cases} 0 & (-a < x < a) \\ \infty & (|x| > a) \end{cases}$$

(a) [8 pts.] Sketch and write down the normalized wave functions for this particle in the ground state, the 1st excited state, and the 2nd excited state.

(b) [6 pts.] If the particle is in the ground state, calculate the probability for finding the particle in the region $a/2 < x < a$, OR $-a/2 < x < 0$.

(c) [6 pts.] Calculate the position and momentum expectation values $\langle x \rangle$ and $\langle p \rangle$ for the 1st excited state.

(d) [5 pts.] Write down an expression involving an integral or integrals, with proper normalization, for the probability that the measured (in the strict quantum mechanical sense) value of $p$ be greater than $\hbar/a$ for the 2nd excited state.
(a) \( y(x) = A \cos b x \)

Normalize it:

\[
\int_{-a}^{a} y^2 \, dx = \int_{-a}^{a} A^2 \cos^2 \left( \frac{bx}{2a} \right) \, dx
\]

\[= 1 \]

\[\rightarrow A = \frac{1}{\sqrt{a}}\]

\[ y_n(x) = \frac{1}{\sqrt{a}} \cos \frac{nx}{2} \quad n = \text{odd} \]

\[ y_n(x) = \frac{1}{\sqrt{a}} \sin \frac{nx}{2} \quad n = \text{even} \]

\[ y_1 = \frac{1}{\sqrt{a}} \cos \left( \frac{\pi x}{2a} \right) \]

\[ y_2 = \frac{1}{\sqrt{a}} \sin \left( \frac{2\pi x}{2a} \right) \]

\[ y_3 = \frac{1}{\sqrt{a}} \cos \left( \frac{3\pi x}{2a} \right) \]

(b) \( P = \int 14^2 \, dx \)

\[ P_1 = \int_{-\frac{a}{2}}^{\frac{a}{2}} 14^2 \, dx \]

\[ = 14 \int_{-\frac{a}{2}}^{\frac{a}{2}} \cos^2 \left( \frac{\pi x}{2a} \right) \, dx \]
\[ \int \cos^2 x \, dx = \frac{x}{2} + \frac{1}{2} \sin x \cos x \]

\[ P_2 = \frac{1}{a} \int_{-a/2}^{a/2} \cos^2 x \, dx = \frac{1}{a} \left[ \frac{1}{2} \frac{\cos x \sin x + x}{2a} \right]_{-a/2}^{a/2} \]

\[ = \frac{2}{a} \left[ 0 + 0 - \frac{1}{2} \frac{\cos \frac{a}{4} \sin \left( -\frac{3a}{4} \right)}{4} + \frac{a}{8} \right] \]

\[ = \frac{1}{4} + \frac{1}{2a} \]

\[ P_1 = \frac{1}{a} \int_{0}^{a} \cos^2 x \, dx = \frac{1}{4} + \frac{1}{2a} \]

\[ P = P_1 + P_2 = \frac{1}{4} + \frac{1}{2a} + \frac{1}{4} - \frac{1}{2a} \]

\[ = \frac{1}{2} \]

Note: This is obvious since \[ \int_{-a/2}^{a/2} \cos^2 x \, dx \]

\[ = \frac{1}{2} \text{ of total probability} \]
\[ \langle x \rangle = \frac{1}{p} \int_{-p}^{p} x \cos^2 \frac{3\pi x}{2a} \, dx = 0 \quad \text{by symmetry} \]

and similarly \[ \langle p \rangle = 0 \quad \text{by symmetry} \]
(a) Let probability for particle to have momentum \( \vec{p} \) in state \( \psi \) be:

\[
\psi_x(\vec{p}) = \int \frac{\phi_1(x) \phi_2(x)}{\sqrt{2a}} \, dx
\]

\( \phi_\mu(x) = A_\mu e^{i \frac{\vec{p} \cdot \vec{x}}{\hbar}} \)

Normalization:

\[
\int_{-\infty}^{\infty} dx \, \bar{\phi}_\mu(x) \phi_\mu(x) = \delta(\vec{p} - \vec{p}')
\]

\[
\therefore \quad \left( \frac{A_\mu}{\sqrt{2a}} \right)^2 \int_{-\infty}^{\infty} e^{-i \frac{\vec{p} \cdot \vec{x}}{\hbar}} \frac{e^{i \frac{\vec{p}' \cdot \vec{x}}{\hbar}}}{\sqrt{2a}} \, dx = \frac{1}{2a} \delta(\vec{p} - \vec{p}')
\]

\[\therefore \quad A_\mu = \frac{\sqrt{\frac{1}{2a}}}{\sqrt{2a}}
\]

\[
\therefore \quad \psi(\vec{p}) = \frac{1}{\sqrt{2a}} \int_{-\infty}^{\infty} e^{-i \frac{\vec{p} \cdot \vec{x}}{\hbar}} = \frac{1}{\sqrt{2a}} \int_{-\frac{\pi}{2a}}^{\frac{\pi}{2a}} e^{-i \frac{\vec{p} \cdot \vec{x}}{\hbar}} \, dx = \frac{1}{\sqrt{2a}} \int_{-\frac{\pi}{2a}}^{\frac{\pi}{2a}} \cos \frac{\vec{p} \cdot \vec{x}}{\hbar} \, dx
\]

\[
= \frac{1}{\sqrt{2a}} \left[ \int_{-\frac{\pi}{2a}}^{\frac{\pi}{2a}} e^{-i \frac{\vec{p} \cdot \vec{x}}{\hbar}} \, dx - \int_{-\frac{\pi}{2a}}^{\frac{\pi}{2a}} \cos \frac{3\vec{p} \cdot \vec{x}}{2a} \, dx \right]
\]

Probability of \( \vec{p} > \frac{h}{2a} \) is:

\[
\int_{\frac{h}{2a}}^{\infty} \psi(\vec{p}) \, d\vec{p}
\]
Problem 3 - Electricity and Magnetism

An electric dipole $\vec{p} = q\vec{d}$ (where vector $\vec{d}$ goes from the negative $-q$ to the positive $+q$ charge of the dipole) is placed at a distance $y$ above an infinite grounded conducting plane. The dipole makes an angle $\theta$ with the $y$-axis. Assume that the dipole can be considered perfect, that is that its size is much smaller than the distance to the conducting plate, $d << y$.

(a) $[15 \text{ pts.}]$ Find the torque $\vec{\tau}$ on $\vec{p}$ (about the center of the dipole).

(b) $[10 \text{ pts.}]$ If the dipole is free to rotate, in what orientation will it come to rest? In other words, what value(s) of $\theta$ represents rotational equilibrium?
Problem 3

(a) Dipole \( \vec{p} \) creates image dipole \( \vec{p}' \) such that \( p'_x = -p_x \)
\( p'_y = p_y \)

Thus \( \vec{p} = p \sin \theta \hat{z} + p \cos \theta \hat{y} \)
\( \vec{p}' = -p \sin \theta \hat{z} + p \cos \theta \hat{y} \)

Torque on dipole \( \vec{p} \) is created by the electric field \( \vec{E}' \) of the image dipole \( \vec{p}' \):

\[ \tau = \vec{p} \times \vec{E}' \]

Note that distance between centers of \( \vec{p} \) and \( \vec{p}' \) is \( 2y \).

Thus we need to work out electric field of the dipole \( \vec{p}' \):

\[ \vec{E}_{\text{dip}} = \left( \frac{q \hat{r}_+}{r_+^2} - \frac{q \hat{r}_-}{r_-^2} \right) \frac{1}{4\pi \varepsilon_0} \]

\[ \vec{r}_+ = \vec{r} - \frac{d}{2} \]
\[ \vec{r}_- = \vec{r} + \frac{d}{2} \]

\[ r_+^2 = (r_+^2)^{3/2} = \left[ \left( \frac{d}{2} - \frac{d}{2} \right) \right]^{3/2} = \left( r^2 + \frac{d^2}{4} - \frac{d^2}{4} \right)^{3/2} = r^3 \left( 1 - \frac{3}{2} \frac{d^2}{r^2} + O \left( \frac{d^4}{r^4} \right) \right) \]

\[ r_-^2 = (r_-^2)^{3/2} = (r_+^2 + \frac{d^2}{4} + \frac{d^2}{4})^{3/2} = r^3 \left( 1 + \frac{3}{2} \frac{d^2}{r^2} \right) \]
$$\vec{E}_{dp} = \frac{q}{4\pi\epsilon_0} \left( \frac{r - \frac{d}{2}}{r^3(1 - \frac{r^2}{2r^2})} - \frac{r + \frac{d}{2}}{r^3(1 + \frac{r^2}{2r^2})} \right)$$

$$= \frac{q}{4\pi\epsilon_0} \cdot \frac{1}{r^3} \left[ (r - \frac{d}{2}) \left( 1 + \frac{3}{2} \frac{r^2}{r^2} \right) - (r + \frac{d}{2}) \left( -\frac{3}{2} \frac{r^2}{r^2} \right) \right]$$

$$= \frac{q}{4\pi\epsilon_0} \cdot \frac{1}{r^3} \left[ -\frac{d}{2} + 3 \frac{r}{r^2} \frac{(\vec{p} \cdot \hat{r})}{r^2} \right] = (\vec{p} = q\vec{d}) =$$

$$= \frac{1}{4\pi\epsilon_0} \cdot \frac{3(\vec{p} \cdot \hat{r})}{r^3} \hat{r} - \vec{p}$$

where $\hat{r} = \text{unit vector}$

$$\vec{C} = \vec{p} \times \vec{E}' = (p_x \vec{i} + p_y \vec{j}) \times \frac{3p_y' \vec{i} - (p_x' \vec{j} + p_y' \vec{i})}{4\pi\epsilon_0 (2y)^3}$$

$$= \frac{1}{4\pi\epsilon_0 (2y)^3} (p_x \vec{i} + p_y \vec{j}) \times (2p_y' \vec{j} - p_x' \vec{i})$$

$$= \frac{1}{4\pi\epsilon_0 (2y)^3} (2p_x' \vec{j} \times \vec{z} - p_y' \vec{i} \times \vec{z})$$

$$= \frac{(2p_x' \vec{j} + p_y' \vec{i}) \vec{z}}{4\pi\epsilon_0 (2y)^3} = \frac{(2p_y - p_y' \vec{j} - p_x' \vec{i}) \vec{z}}{4\pi\epsilon_0 (2y)^3}$$

$$= \frac{p^2 \sin(2\theta) \vec{k}}{4\pi\epsilon_0 (2y)^3} = \frac{p^2 \sin(2\theta) \vec{k}}{4\pi\epsilon_0 \cdot 16y^3}$$
(b) The torque $\tau \sim \sin(2\theta)$ [from part (a)]

$\Rightarrow \tau = 0 \text{ when } \sin(2\theta) = 0$

$2\theta = n\pi$

$\theta = \frac{n\pi}{2} = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

Out of these 4 positions, only $\theta = 0$ or $\pi$ are stable equilibrium positions.

$\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ represent unstable equilibrium.

This follows from considering the energy of dipole $\vec{p}$ in the electric field of $\vec{p}'$:

\[ U = -\vec{p} \cdot \vec{E}' = -(p_x x' + p_y y'). \frac{(2p' y^2 - p x^2)}{4\pi \varepsilon_0 (2y)^3} = \]

\[ = -\frac{1}{4\pi \varepsilon_0 (2y)^3} \left( 2 p y y' - p x p' \right) = \]

\[ = \frac{-1}{4\pi \varepsilon_0 (2y)^3} \left( 2 p y^2 + p x^2 \right) = -\frac{p^2 (\sin^2 \theta + 2 \cos^2 \theta)}{4\pi \varepsilon_0 (2y)^3} = \]

\[ = -\frac{1}{4\pi \varepsilon_0} \frac{p^2}{8y^3} (1 + \cos^2 \theta) \]

Equilibrium positions are $\frac{dU}{d\theta} = 0 \Rightarrow \sin(2\theta) = 0$ as before.

The minimum is achieved for $\frac{d^2U}{d\theta^2} > 0$, that is

when $\cos(2\theta) > 0$

$\Rightarrow \theta = 0, \pi$ are minimum energy positions

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ are maximum energy positions.
Problem 4 - Statistical Mechanics

A container of xenon gas (Xe) contains approximately $8 \times 10^{22}$ atoms. The atoms are not under the influence of outside forces. Neglect any thermal effects of the container. The ionization energy (for loss of 1 electron) is 12.13 eV. (Numerical answers required.)

(a) [5 pts.] Find the approximate number of these gas atoms which are ionized at an equilibrium temperature of 3000 degrees K.

(b) [5 pts.] Find the low-temperature heat capacity (at constant volume) ($C_{\text{gas}}$) of $8 \times 10^{22}$ Xe gas atoms (Joules/deg K).

(c) [5 pts.] The container of $8 \times 10^{22}$ Xe gas atoms at a temperature of 202 degrees K is brought in thermal contact with a large heat reservoir at a temperature of 200 degrees K. Determine the change in the entropy of the reservoir after the two are brought into thermal contact, and have reached thermal equilibrium. You may assume that the specific heat capacity of the Xe gas has the value found in part b.

(d) [10 pts.] If the Xenon gas atoms (Xe mass m) are confined to move in the $\hat{x}$ direction (you may completely neglect the other two dimensions), at an equilibrium temperature $T$, find the normalized probability distribution $f(v_x)$ of the Xenon gas atoms. This probability distribution describes the probability density that a given atom has a velocity $v_x$. (Put your answer in terms of the mass m, Boltzmann’s constant k, temperature T.)

1 eV = $1.6 \times 10^{-19}$ joules
Boltzmann’s constant: $k = 1.381 \times 10^{-23}$ joules/K
Problem 4

8 x 10^{22} \text{ Xenon atoms}

\text{Ionization energy = 12.13 eV}

a) \[ \frac{N_{\text{ionized}}}{N_{\text{total}}} = e^{-\frac{E_{\text{ion}}}{kT}} \]

\[ E_{\text{ion}} = (12.13 \text{ eV}) \left( \frac{1.6 \times 10^{-19} \text{ J}}{1 \text{ eV}} \right) = 1.94 \times 10^{-18} \text{ J} \]

\[ kT = 4.14 \times 10^{-20} \]

\[ T = 3000 \text{ K} \]

\[ N_{\text{ionized}} = N_{\text{total}} e^{-\frac{E_{\text{ion}}}{kT}} \]

\[ N_{\text{ionized}} = (8 \times 10^{22}) e^{-46.8} \approx 363 \]

b) At low temperatures

3 translational degrees of freedom

\[ E = \frac{3kT}{2} \text{ / atom} \]

\[ C_{\text{atom}} = \frac{\Delta E}{\Delta T} = \frac{3}{2} k \]

\[ C_{\text{gas}} = N_{\text{total}} C_{\text{atom}} = \frac{3}{2} k N_{\text{total}} = 1.66 \text{ Joules/°K} \]
C) Total heat flow during thermal contact = $\Delta Q$

Temperature change of the gas = $\Delta T_c = 2^\circ K$

$\Delta Q = C_{gas} \Delta T_{gas} = (1.66 \ T/K)(2^\circ K) = 3.31 \ Joules$

Entropy change of the reservoir:

$\Delta S = \frac{\Delta Q}{T_{res}} = \frac{3.31 \ Joules}{202^\circ K} = 0.0166 \ Joules/K$

d) $f(v_x) = C \ e^{-\frac{E}{kT}} = C \ e^{-\frac{mv_x^2}{2kT}}$

Find normalization constant $C$

$\int_{-\infty}^{\infty} C \ e^{-\frac{mv_x^2}{2kT}} \ dv_x = 1$

$C \int_{-\infty}^{\infty} e^{-\frac{mv_x^2}{2kT}} \ dv_x = C (\pi)^{\frac{1}{2}} (\frac{m}{kT})^{-\frac{1}{2}} = 1$

$C = \left( \frac{m}{2\pi kT} \right)^{\frac{1}{2}}$

$f(v_x) = \left( \frac{m}{2\pi kT} \right)^{\frac{1}{2}} e^{-\frac{mv_x^2}{2kT}}$
Problem 5 - Modern Physics

A typical fixed target particle physics experiment from the 1980’s is depicted in the diagram. A primary beam of protons strikes an iron production target. The collision produces many particles. In some cases, an unstable, but energetic particle may emerge from the target, only to decay downstream of the target.

In the “event” shown, a neutral parent particle, X, decays into a pair of singly-charged daughter particles. First, the decay “vertex” is identified using a silicon vertex tracker, which also measured the opening angle, $\theta$, between the daughter particles. Next, the momenta, $p_1$ and $p_2$, of the daughter particles are determined using a magnetic spectrometer. Drift chambers track the trajectories of the particles and measure their curvature in a uniform magnetic field. In this particular event, the trajectories of the particles are confined to within a plane perpendicular to the magnetic field. The magnetic field in the figure points out of the page. Finally, the total energies $E_1$ and $E_2$ (including rest mass energies) of the daughter particles are measured using a segmented calorimeter, which typically comprise blocks of NaI or CsI scintillators, or lead glass, each coupled to photo-multiplier tubes. (Numerical answers are required.)

**Data:**

- Speed of Light $c = 3.00 \times 10^8$ m/s
- Elementary charge $e = 1.60 \times 10^{-19}$ C
- Particle 1: $E_1 = 4.66$ GeV, $R_1 = 75.0$ m (radius of curvature of particle 1 in the B-field)
- Particle 2: $E_2 = 2.53$ GeV, $R_2 = 40.3$ m (radius of curvature of particle 2 in the B-field)
- Opening angle $\theta = 12.0^\circ$ ; $B = 0.200$ tesla ; $1$ GeV $= 1.60 \times 10^{-10}$ J

(a) [2 pts.] What are the signs of the charges of the two particles (labeled 1 and 2)?

(b) [8 pts.] Show that the radius of curvature $R$ of a RELATIVISTIC particle of charge $e$, moving in uniform circular motion perpendicular to a uniform magnetic field $B$, is related to its momentum by $p = bR$. Find the value of $b$ in units of (GeV/c)/m for this experiment.

If you cannot solve part (b) assume a value of $b = 0.0500$ (GeV/c)/m and state so if you use this value.

(c) [7 pts.] Find the mass, $m_1$ and $m_2$, of the two daughter particles in units of GeV/c$^2$.

(d) [8 pts.] Find the mass, $M_X$, of the parent particle X.
Problem 5

(a) \[ \vec{E} = g \vec{V} \times \vec{B} \]

Since \( \vec{B} \) points out of the page, \( \vec{V} \times \vec{B} \) tends to deflect the particle to the "right."

\[ \text{i.e.:} \quad \vec{B} \rightarrow \vec{V} \]
\[ \downarrow \vec{V} \times \vec{B} \]

Obviously then \( g_2 > 0 \) and \( g_1 < 0 \)

i.e: particle 1 is negative
\[ i \] is positive

(b) RELATIVISTIC motion in a circle

Since we have uniform circular motion, we have \( V = \text{constant} \)

and \( |\vec{a}| = |\frac{d\vec{V}}{dt}| = \frac{V^2}{R} \) (regardless of whether particle is relativistic or not)

Newton's law: \( \frac{d\vec{F}}{dt} = \vec{F}_m \)

\[ \vec{F} = \gamma m \vec{V} \]
\[ \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \]

Note that \( \gamma = \text{constant for uniform circular motion} \)

So \( \frac{d\vec{F}}{dt} = \gamma m \frac{d\vec{V}}{dt} \)

In this case, at the moment shown
\[ \vec{V} = V' \]

and \( \frac{d\vec{F}}{dt} = \gamma m \left( \frac{dv_x}{dt} \hat{i} + \frac{dv_y}{dt} \hat{j} \right) = g V B \hat{i} \)
(b) cont'd here we identify \( \frac{dV_x}{dt} = a = \frac{v^2}{R} \) as the "centripetal acceleration".

\[ \Rightarrow \frac{dv_x}{dt} = 0 \] at the moment shown

and \( \frac{\dot{y} m}{dt} = \dot{y} m a = \frac{\dot{y} m v^2}{R} = q v B \)

\[ \Rightarrow \frac{\dot{y} m v}{q B R} \]

\[ \Rightarrow P = \frac{(q B) R}{(eB) R} \]

\[ \Rightarrow P = \frac{(eB) R}{b R} = \frac{b}{e} \]

\[ b = (1.6 \times 10^{-19} \text{ C})(0.2 \text{G}) = 3.2 \times 10^{-20} \text{ C} \cdot T = 3.2 \times 10^{-20} \frac{nC}{m} \]

\[ = 3.2 \times 10^{-20} \left( \frac{1 \text{ GeV}}{1.6 \times 10^{-10} \text{ J}} \right) \frac{3.0 \times 10^8 \text{m/s}}{c} = 0.0600 \text{ GeV/c} \]

Note that the formula

\[ P = (eB) R \]

is the same for non-relativistic particles

where \( \gamma \to 1 \)
(c) \[ P_1 = b \hat{R}_1 = (0.0600 \ \text{GeV/c}) (75.0 \ \text{m}) = 4.50 \ \text{GeV/c} \]
\[ E_1 = 4.66 \ \text{GeV} \]
For relativistic particles:
\[ E^2 = m^2 c^4 + p^2 c^2 \Rightarrow m = \frac{1}{c^2} \sqrt{E^2 - p^2 c^2} \]
\[ m_1 = \frac{1}{c^2} \sqrt{(4.66 \ \text{GeV})^2 - (4.50 \ \text{GeV/c})^2} c^2 \]
\[ = \frac{1}{c^2} \sqrt{1.4656 \ \text{GeV}^2} \]
\[ m_1 = 1.21 \ \text{GeV/c}^2 \]

Using \( b = 0.05 \ \text{GeV/c} \)
\[ m_1 = 2.77 \ \text{GeV/c}^2 \]

\[ P_2 = b \hat{R}_2 = (0.0600 \ \text{GeV/c}) (40.3 \ \text{m}) = 2.418 \ \text{GeV/c} \]
\[ m_2 = \frac{1}{c^2} \sqrt{(2.53 \ \text{GeV})^2 - (2.418 \ \text{GeV/c})^2} c^2 \]
\[ = \frac{1}{c^2} \sqrt{0.5542 \ \text{GeV}^2} \]
\[ m_2 = 0.744 \ \text{GeV/c}^2 \]

Using \( b = 0.05 \ \text{GeV/c} \)
\[ m_2 = 1.53 \ \text{GeV/c}^2 \]

(d) Part e is really easy, with 4-vectors!
\[ \begin{align*}
\{ P_1 & = (E_1/c, \vec{P}_1) \\
\{ P_2 & = (E_1/c, \vec{P}_2) \}
\end{align*} \]
4-vectors of the daughter particles right after the decay.

By conservation of energy and momentum, we have \( P_x = (E_x, P_x) = P_1 + P_2 \)
\[ \text{momentum 4-vector of X before decay}. \]
Problem 5

(a) cont'd.

⇒ \( P_x = \left( \frac{E_1}{c} + \frac{E_2}{c}, \frac{P_1}{c} + \frac{P_2}{c} \right) \)

but \( M_x^2 = \frac{1}{c^2} (P_x)^2 = \left( \frac{E_1^2}{c^2} - \frac{P_x \cdot P_x}{c^2} \right) \)

⇒ \( \frac{1}{c^2} \left[ \frac{1}{c^2} (E_1 + E_2)^2 - \left( \frac{P_1}{c} + \frac{P_2}{c} + 2 \frac{P_1 \cdot P_2}{c^2} \right) \right] \)

but \( P_1 \cdot P_2 = P_1 P_2 \cos \theta \)

Opening angle measured by the silicon vertex detector

⇒ \( M_x = \frac{1}{c^2} \sqrt{\left( E_1 + E_2 \right)^2 - c^2 \left( P_1^2 + P_2^2 + 2 P_1 \cdot P_2 \cos \theta \right)} \)

⇒ \( M_x = \frac{1}{c^2} \sqrt{\left( 4.60 + 2.53 \right)^2 \text{GeV}^2 - c^2 \left[ (4.50)^2 + (2.418)^2 + 2(4.50)(2.418) \cos(2\theta) \right] \text{GeV}^2} \)

⇒ \( M_x = \frac{1}{c^2} \sqrt{51.70 \text{ GeV}^2 - 47.38 \text{ GeV}^2} \)

⇒ \( M_x = 2.08 \text{ GeV}/c^2 \)

⇒ \( M_x = 4.53 \text{ GeV}/c^2 \)

Using \( b = 0.05 \text{ GeV}/c^2 \)

⇒ \( M_x = 4.53 \text{ GeV}/c^2 \)

Alternatively, doing this problem without 4-vectors is a little trickier: I will do it this way over the next 2 pages.
Problem 5

Assuming the parent particle started with energy $E_X$ and momentum $P_X$ in the $x$-direction.

\[ E_X = \sqrt{m^2 c^4 + P_X^2 c^2} \]

\[ \Theta_1 + \Theta_2 = \Theta = 120^\circ \]

Assume these are "positive" angles.

Conservation of momentum:
\[ P_X = P_1 \cos \Theta_1 + P_2 \cos \Theta_2 \quad \ldots (1) \quad (x\text{-dir.}) \]
\[ 0 = P_1 \sin \Theta_1 - P_2 \sin \Theta_2 \quad \ldots (2) \quad (y\text{-dir.}) \]

Conservation of energy:
\[ \sqrt{M^2 c^4 + P_X^2 c^2} = \sqrt{m_1^2 c^4 + P_1^2 c^2} + \sqrt{m_2^2 c^4 + P_2^2 c^2} \quad \ldots (3) \]

First we square both sides of (3):
\[ M^2 c^4 + P_X^2 c^2 = m_1^2 c^4 + P_1^2 c^2 + m_2^2 c^4 + P_2^2 c^2 \]
\[ + 2 \sqrt{m_1^2 c^4 + P_1^2 c^2} \sqrt{m_2^2 c^4 + P_2^2 c^2} \quad \ldots (4) \]

Next we take $(1)^2 + (2)^2$:
\[ P_X^2 = P_1^2 \cos^2 \Theta_1 + P_2^2 \cos^2 \Theta_2 + 2P_1P_2 \cos \Theta_1 \cos \Theta_2 \]
\[ + P_1^2 \sin^2 \Theta_1 + P_2^2 \sin^2 \Theta_2 - 2P_1P_2 \sin \Theta_1 \sin \Theta_2 \]
\[ = P_1^2 + P_2^2 + 2P_1P_2 \cos (\Theta_1 + \Theta_2) \quad \ldots (5) \]

\[ \Theta_1 + \Theta_2 = \Theta \]

Note:
\[ \cos \Theta_1 \cos \Theta_2 - \sin \Theta_1 \sin \Theta_2 = \cos (\Theta_1 + \Theta_2) \]
Next: Substitute (5) into the LHS of (4) and observe that \( \sqrt{m_1 c^4 + p_1^2 c^2} \cdot \sqrt{m_2 c^4 + p_2^2 c^2} = E_1 E_2 \) for the RHS of (4).

\[
\Rightarrow M^2 c^4 + p_1^2 c^2 + p_2^2 c^2 + 2p_1 p_2 \cos \theta c^2 = m_1 c^4 + p_1^2 c^2 + m_2 c^4 + p_2^2 c^2 + 2 E_1 E_2 \]

but \( m_1 c^4 = E_1^2 - p_1^2 c^2 \)
\( m_2 c^4 = E_2^2 - p_2^2 c^2 \)

\[
\Rightarrow M^2 c^4 = E_1^2 - p_1^2 c^2 + E_2^2 - p_2^2 c^2 + 2 E_1 E_2 - 2p_1 p_2 \cos \theta c^2 \]

\[
= (E_1 + E_2)^2 - c^2 (P_1^2 + P_2^2 + 2P_1 P_2 \cos \theta) \]

\[
M = \frac{1}{c^2} \sqrt{(E_1 + E_2)^2 - c^2 (P_1^2 + P_2^2 + 2P_1 P_2 \cos \theta)} \]

Which is the same equation as we used on p. 4.
Examination booklets have been provided for recording your work and your solutions. *Please note that there is a separate booklet for each numbered question (i.e., use booklet #1 for problem #1, etc.).*

To receive full credit, not only should the correct solutions be given, but a sufficient number of steps should be given so that a faculty grader can follow your reasoning. Define all algebraic symbols that you introduce. If you are short of time it may be helpful to give a clear outline of the steps you intended to complete to reach a solution. In some of the questions with multiple parts you will need the answer to an earlier part in order to work a later part. If you fail to solve the earlier part you may represent its answer with an algebraic symbol and proceed to give an algebraic answer to the later part. This is a closed book exam: No notes, books, or other records should be consulted. YOU MAY ONLY USE THE CALCULATORS PROVIDED. The total of 250 points is divided equally among the ten questions of the examination.

All work done on scratch paper should be NEATLY transferred to answer booklets.

SESSION 2
Problem 6 - General Physics

A solid wooden cylinder of density $S = 0.700 \, \text{g/cm}^3$ and radius $R = 5.00 \, \text{cm}$ is partially immersed in a bath of distilled water. The length of the cylinder is $h = 20.0 \, \text{cm}$. The cylinder is pushed down further into the water from its equilibrium, floating position slightly and then released. The cylinder then undergoes simple vertical harmonic oscillations. The density of water is $S_w = 1.00 \, \text{g/cm}^3$. (Numerical answers required.)

(a) [5 pts.] Draw a free-body diagram (also known as force diagram) for the system. This diagram should indicate all forces acting on the cylinder.

(b) [5 pts.] Find the distance between the water level and the top of the cylinder when it is in its equilibrium, floating position.

(c) [5 pts.] Find the natural frequency, in hertz, of small, vertical oscillation of the cylinder, assuming it stays upright at all times. Neglect friction.

(d) [5 pts.] What is the frequency of small vertical oscillations of the cylinder if salt were added to the water to change its density to $S_w' = 1.20 \, \text{g/cm}^3$.

(e) [5 pts.] Identify and describe two sources of energy dissipation in this system.
Problem 6

(i) \[ F_B \]

(ii) \[ h - x = ? \]

\[ \pi r^2 x g \delta \omega = \pi r^2 h g \delta \omega \]

\[ \frac{x}{h} = \frac{g \delta \omega}{g \delta} \]

\[ x = 0.7 \times 20 \times \frac{1}{1} = 14 \text{ cm} \]

\[ h - x = 6 \text{ cm} \]

(iii) \[ \Sigma F_x = ma \]

Weight of water displaced = \( (\pi r^2 x) g \delta \omega \]

= restoring force

Mass of cylinder = \( (\pi r^2 h g) \]

\[ \therefore m \ddot{x} + f \dot{x} = 0 \]

\[ T \pi r^2 h S \ddot{x} + S \pi r^2 g x = 0 \]

\[ \omega = \sqrt{\frac{g}{h S}} \]

\[ f = 0.33 \text{ Hz} \]

(iv) With salt in water restoring force is higher
Problem 6

\[ 1.2 \left( \frac{5}{\pi} \cdot n^2 \cdot \pi^2 \cdot x \right) g. \]

\[ \therefore \pi n^2 \Delta S \ddot{x} + 1.2 \left( \frac{5}{\pi} \cdot n^2 \cdot x \right) \ddot{x} = 0. \]

\[ \omega = \sqrt{\frac{1.2 g}{\pi \Delta S}} \]

i.e. frequency is increased as restoring force is increased.

(5) energy dissipation = viscous losses in fluid + sound radiated by oscillating cylinder.
Problem 7 - Thermodynamics

A particular motorcycle engine uses an ideal gas as its working fluid in a sealed cylinder, and has a 3-phase cycle as shown in the diagram:

A: The cylinder is pressurized from \((P_1, V_1, T_1)\) to \(P_2\) at constant volume.
B: The cylinder undergoes adiabatic expansion from \((P_2, V_2)\) to \((P_3, V_3)\)
C: The system is allowed to deflate at constant pressure to its original state \((P_1, V_1, T_1)\).

Data:

\[
V_1 = V_2 = 20.00 \text{ cm}^3, \quad V_3 = 170.0 \text{ cm}^3 \\
(\text{this is a 170 cc engine with a 8.5:1 compression ratio}). \\
T_1 = 400.0 \text{ K}, \quad P_1 = P_3 = 150.0 \text{ kPa}, \quad P_2 = 3001 \text{ kPa}. \\
\text{Molar Gas Constant } R = 8.314 \text{ J K}^{-1} \text{ mol}^{-1}.
\]

(Numerical answers are required.)

(a) [3 pts.] Calculate the number of moles of gas present in the sealed cylinder.

(b) [4 pts.] Find the temperatures \(T_2\) and \(T_3\) in kelvins.

(c) [5 pts.] Assuming that vibrational degrees of freedom are completely frozen out, but that the rotational degrees of freedom are fully active, use the data given to determine whether the working ideal gas is made of mono-atomic, di-atomic, or multi-atomic (more than 2 atoms) molecules. You must give a numerical analysis to receive any credit. A correct answer without any supporting work will receive zero credit.

The answers to the remaining parts depend on your answer to part (c). If you cannot answer part (c), then assume the gas to be helium for the remaining parts. But you must then state clearly that you are making this assumption.

(d) [4 pts.] Find the work done by the engine, in joules, during each phase of the cycle.

(e) [6 pts.] Find the heat flowing into the engine, in joules, during each phase of the cycle.

(f) [3 pts.] Give the appropriate definition for \(\eta\), the efficiency of the engine, in the context of the engine, and determine its value based on the data and your results from previous parts.
ideal gas equation of state: $PV = nRT$

$n = \# \text{ of moles present} : \text{use part (i): } (P_1, V_1, T_1)$

$$n = \frac{PV}{RT} = \frac{P_1V_1}{RT_1} = \frac{(1.50 \times 10^5 \text{Pa}) (2000 \text{ cm}^3)}{(8.31 \text{ J K}^{-1} \text{mol}^{-1}) (400 \text{ K})}$$

$$n = 9.021 \times 10^{-4} \text{ mol}$$

(b) The cylinder is sealed $\Rightarrow n = \text{constant}$.

$$\Rightarrow \frac{PV}{T} = \text{constant}$$

$$\frac{P_2V_2}{T_2} = \frac{P_1V_1}{T_1} \Rightarrow T_2 = \frac{P_2V_2T_1}{P_1V_1} = \frac{(300 \text{ kPa})(400 \text{ K})}{150 \text{ kPa}}$$

$$T_2 = 800.3 \text{ K}$$

Similarly $T_3 = \frac{P_3V_3T_1}{P_1V_1} = \frac{170 \text{ cm}^3}{20 \text{ cm}^3} (400 \text{ K})$

$$T_3 = 8400 \text{ K}$$

(c) One can tell the type of gas from the adiabatic factor $\gamma$, which is also the ratio of $C_p/C_v = \gamma$.

During adiabatic processes: we have $PV^\gamma = \text{constant}$

$$\Rightarrow P_2V_2^\gamma = P_3V_3^\gamma \Rightarrow \left(\frac{V_3}{V_2}\right)^\gamma = \left(\frac{P_3}{P_2}\right)$$

$$\Rightarrow \gamma \ln \left(\frac{V_3}{V_2}\right) = \ln \left(\frac{P_3}{P_2}\right) \Rightarrow \gamma = \frac{\ln \left(\frac{P_3}{P_2}\right)}{\ln \left(\frac{V_3}{V_2}\right)} = \frac{\ln \left(\frac{300 \text{ kPa}}{150 \text{ kPa}}\right)}{\ln \left(\frac{170 \text{ cm}^3}{20 \text{ cm}^3}\right)}$$

--- Cont'd
(c) cont'd

\[ y = \frac{\ln(20.01)}{\ln(1.50)} = \frac{2.996}{2.140} = 1.400 \]

Recall for ideal gases: \( C_p = C_V + R \) (per mole)

\[ C_p = \frac{D}{2} R \]
\[ C_V = \frac{D-1}{2} R \]

\[ \frac{C_p}{C_V} = \frac{D}{D-1} = \frac{D+2}{D} \]

Here \( \frac{D+2}{D} = 1.4 \Rightarrow D+2 = 1.4D \Rightarrow 0.4D = 2 \]

\[ \Rightarrow D = 5 \]

di-atomic gases have 3 translational + 2 rotational degrees of freedom

\[ \Rightarrow \text{working gas is diatomic} \]

(a) For work done by the system:

\[ dW = PdV \] (note the sign convention)

phase A: \( \Delta V = 0 \Rightarrow W_A = 0 \)

B: \[ W_B = \int_{V_2}^{V_3} P \, dV \]

Here: \( PV^r = \text{constant} \Rightarrow PV^r = P_2V_2^r \)

or: \( P = P_2\left(\frac{V_2}{V}\right)^r = P_2V_2^r V^{-r} \)

\[ W_B = \int_{V_2}^{V_3} (P_2V_2^r) V^{-r} \, dV = \frac{-1}{r-1} (P_2V_2^r) V^{-(r-1)} \bigg|_{V_2}^{V_3} \]
(d) \( W_B = \frac{1}{\gamma - 1} (P_2 V_2^\gamma) \left( \frac{1}{V_2} - \frac{1}{V_3} \right) \)

\[
= \left( \frac{1}{0.4} \right) \left( 3.001 \times 10^6 \frac{N}{m^2} \right) \cdot \left( 2.0 \times 10^{-5} \ m^3 \right)^{1.4} \left[ \frac{1}{2.0 \times 10^{-5} \ m^3} \cdot \frac{1}{(1.7 \times 10^{-4} \ m^2)^{0.4}} \right] \\
= (7.507 \times 10^6 \frac{N}{m^2}) \left( 2.639 \times 10^{-7} \ m^4 \cdot 2 \right) \\
x \left[ 75.79 \ m^{-1.2} - 32.20 \ m^{-1.2} \right] \\
W_B = 86.31 \ J
\]

Note: Using \( \frac{1}{\gamma - 1} (P_2 V_2^\gamma) \left( \frac{1}{V_2} - \frac{1}{V_3} \right) \)

C: Constant \( P = P_1 \)

\[ W_C = \int_{V_3}^{V_1} P_1 \ dV = P_1 (V_1 - V_3) = (1.5 \times 10^5 \frac{N}{m^2}) (2.0 \times 10^{-5} - 1.7 \times 10^{-4}) m^2 \]

\[ W_C = -22.50 \ J \]

(e) \( A: \) Constant Volume

\[ Q_A = hC V \Delta T = hC (T_2 - T_1) \]

\[ = h \left( \frac{p}{\gamma} \right) R (T_1 - T_i) \]

\[ = (9.021 \times 10^{-4} \ mol) \left( \frac{8.314 \ J \cdot K^{-1} \cdot mol^{-1}}{2} \right) (8.314 \ J \cdot K^{-1} \cdot mol^{-1}) (8003 - 400) \ K \]

\[ Q_A = 142.56 \ J \]
(e) cont'd:

\[ Q_B = 0 \quad \text{for adiabatic expansion} \]

\[
Q_c = nC_p \Delta T = n \left( \frac{\beta}{2} + 1 \right) R \left( T_1 - T_2 \right) \quad \text{(constant pressure)}
\]

\[
= (9.021 \times 10^{-4} \text{ mol}) \left( \frac{\beta}{2} \right) (8.314 \text{ JK}^{-1}\text{mol}^{-1})(400 - 340) \text{ K}
\]

\[ Q_c = -78.75 \text{ J} \]

(f) \[ \eta = \frac{W_{\text{net}}}{Q_{\text{in}}} \]

where \( W_{\text{net}} = \) net work done by the system

\[ = W_A + W_b + W_c = +63.81 \text{ J} \]

and \( Q_{\text{in}} = Q_A \) (only the heat put into the engine; \( Q \) is heat "loss")

\[ Q_{\text{in}} = +142.56 \text{ J} \]

\[ \eta = \frac{63.81}{142.56} = 44.76\% \]

Note by conservation of energy, we'd expect \( Q_A + Q_b \)

\[ W_A + W_b + W_c = 63.81 \text{ J} \]

and indeed \( Q_b = 0 \) and \( Q_A + Q_b = +63.81 \text{ J} \)!

So we could also express \( \eta \) as \( (Q_c \text{ is negative!}) \)

\[ \eta = \frac{Q_{\text{in}} - Q_{\text{out}}}{Q_{\text{in}}} = \frac{Q_A + Q_c}{Q_A} = \boxed{44.76\%} \]

also valid.
(a) If we assume the gas is helium:
then we cannot actually get a consistent result.

But here are some possibilities:
for helium, which is mono-atomic, \( D = 3 \), \( \gamma = \frac{5}{3} \)
we again have \( W_a = 0 \)
and \( W_b' = P_1 (V_1 - V_\beta) = -22.50 J \)
But \( W_c' = \frac{1}{\gamma - 1} P_2 V_2 \gamma \left( \frac{1}{V_2^\gamma} - \frac{1}{V_\beta^\gamma} \right) \)
Using \( P_2, V_2 \)

\[
W_c' = \frac{1}{0.667} \left( \frac{3.00 \times 10^{6} N}{m^2} \right) \left( 2.0 \times 10^{-5} m^3 \right)^{\frac{5}{3}} \times \left[ \frac{1}{(2.0 \times 10^{-5})^{0.667}} - \frac{1}{(1.70 \times 10^{-4})^{0.667}} \right] m^{-2}
\]

\[
W_c' = +68.41 J
\]

OR one could use \( (P_2, V_\beta) \) as the reference point

\[
W_c'' = \frac{1}{0.667} \left( 1.50 \times 10^{5} N/m^2 \right) \left( 1.70 \times 10^{-4} m^3 \right)^{1.667} \times \left[ \frac{1}{(2.0 \times 10^{-5})^{0.667}} - \frac{1}{(1.70 \times 10^{-4})^{0.667}} \right] m^{-2}
\]

\[
W_c'' = 121.1 J
\]

Note the two do not give the same answer
(it did in the correct \( \gamma = 1.4 \) case)
(e) a. Heats

\[ Q_A = nC_v \Delta T = n \left( \frac{D}{2} \right) R (T_2 - T_1) \]
\[ = (9.021 \times 10^{-4} \text{ mol}) \left( \frac{D}{2} \right) (8.314 \text{ J K}^{-1} \text{ mol}^{-1}) (8005 - 400 \text{ K}) \]
\[ Q_A = 85.53 \text{ J} \]
\[ Q_B = 0 \]
\[ Q_C' = nC_p \Delta T = n \left( \frac{D}{2} + 1 \right) R (T_1 - T_2) \]
\[ = (9.021 \times 10^{-4} \text{ mol}) \left( \frac{D}{2} + 1 \right) (8.314 \text{ J K}^{-1} \text{ mol}^{-1}) (400 - 3400 \text{ K}) \]
\[ Q_C' = -56.25 \text{ J} \]

(f) 3 possible answers for \( \eta' \) based on what we have

(i) \[ \eta' = \frac{W_A' + W_B' + W_C'}{Q_A'} = \frac{45.91 \text{ J}}{85.53 \text{ J}} = 0.5368 \text{ %} \]

(ii) \[ \eta' = \frac{W_A'' + W_B'' + W_C''}{Q_A'} = \frac{98.60 \text{ J}}{85.53 \text{ J}} = 115.3 \text{ %} \]

(iii) \[ \eta' = \frac{Q_A' + Q_C'}{Q_A'} = \frac{24.28 \text{ J}}{85.53 \text{ J}} = 28.28 \text{ %} \]
Problem 8 - Mechanics

Consider a pendulum (point mass $m_2$ connected to a massless rod of length $l$) whose support point with a point mass $m_1$ in it, is free to move along the horizontal direction as shown in the figure.

![Diagram of a pendulum with a mass $m_1$ at the bottom and a mass $m_2$ at the end of a rod, with a support point at the bottom of the rod that is free to move horizontally.]

The “sliding” pendulum - support point of mass $m_1$ is free to move.

(a) [2 pts.] Write down the Cartesian/Rectilinear coordinates $(x_1, y_1)$ and $(x_2, y_2)$ of the masses $m_1$ and $m_2$ in the coordinate system with $y$-axis pointing down, as shown in the figure, in terms of $x$ and angle $\phi$ between the rod and the $y$-axis.

(b) [4 pts.] Derive the Lagrangian $L = T - U$ for this system in terms of $x$, $\phi$, $\dot{x}$ and $\dot{\phi}$.

(c) [2 pts.] Observe that the coordinate $x$ does not show up explicitly in the Lagrangian. Show, using the equation of motion, that this implies that the $x$-component of the generalized momentum is conserved,

$$P_x = \frac{\partial L}{\partial \dot{x}}$$

Derive an expression for $P_x$.

(d) [5 pts.] Choose $P_x = 0$ (this always can be done due to Galilean invariance) and integrate the equation in (c) over time. What is the physical meaning of the obtained equation?

(e) [8 pts.] Now, using your result from (c), and again choosing $P_x = 0$, express the Lagrangian in terms of the angle $\phi$ (and its derivative) only. Consider small (i.e., harmonic) oscillations around the minimum $\phi_0 = 0$ and derive an expression for the frequency $\omega$ of such oscillations.

(f) [4 pts.] Analyze the limit of your result in (e) for the frequency when $m_1 \to \infty$. What does it correspond to? What happens if the opposite limit, $m_1 \to 0$, is taken, and why?
(a) \[
\begin{align*}
\dot{x}_1 &= x, \\
\dot{y}_1 &= 0,
\end{align*}
\begin{align*}
\dot{x}_2 &= x + \ell \cos \theta, \\
\dot{y}_2 &= \ell \sin \theta.
\end{align*}
\]

(b) \[
\mathbf{T} = T - U, \quad T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)
\]

(c) \[
\mathbf{L} = -m_2 y \dot{y}_2
\]

\[
\Rightarrow T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \left[ (\dot{x} + \ell \dot{\theta} \cos \theta)^2 + (\ell \dot{\theta} \sin \theta)^2 \right] = \frac{1}{2} (m_1 + m_2) \dot{x}^2 + \frac{1}{2} m_2 \left( \ell^2 \dot{\theta}^2 + 2 \ell \dot{x} \dot{\theta} \cos \theta \right)
\]

\[
\Rightarrow L = \frac{1}{2} (m_1 + m_2) \dot{x}^2 + \frac{1}{2} m_2 \left( \ell^2 \dot{\theta}^2 + 2 \ell \dot{x} \dot{\theta} \cos \theta \right) + m_2 \ell \dot{\theta} \cos \theta
\]

(c) \[
\frac{\partial L}{\partial \dot{x}} = 0 \Rightarrow \frac{\partial}{\partial \dot{x}} \frac{\partial L}{\partial \dot{x}} = 0 \Rightarrow P_x = \frac{\partial L}{\partial \dot{x}} = \text{const.}
\]

\[
P_x = (m_1 + m_2) \dot{x} + m_2 \ell \dot{\theta} \cos \theta = \frac{d}{dt} \left[ (m_1 + m_2) x + m_2 \ell \sin \theta \right]
\]

\[
= \frac{d}{dt} \left[ m_1 x + m_2 (x + \ell \sin \theta) \right]
\]

\[
\Rightarrow P_x \text{ is the } x\text{-component of total momentum.}
\]

(d) \[
P_x = 0 \Rightarrow m_1 x + m_2 (x + \ell \sin \theta) = 0
\]

This is an expression for the center of mass of the system.

Now, \[
P_x = 0 \Rightarrow \dot{x} = -\frac{m_2 \ell \dot{\theta} \cos \theta}{m_1 + m_2}
\]
}\Rightarrow L = \frac{1}{2} \left( m_1 + m_2 \right) \left( \frac{m_2 l^2 \omega^2 \sin^2 \theta}{m_1 + m_2} \right) + \frac{1}{2} m_2 l^2 \dot{\theta}^2 + \nonumber \\
+ m_2 l \left( \frac{m_2 l \dot{\theta} \omega \sin \theta}{m_1 + m_2} \right) \dot{\theta} \omega \sin \theta + m_2 g l \cos \theta 

= \frac{1}{2} m_2 l^2 \ddot{\theta}^2 \left( 1 - \frac{m_2}{m_1 + m_2} \cos^2 \theta \right) + m_2 g l \cos \theta 

= \frac{1}{2} \frac{m_1 m_2 l^2}{m_1 + m_2} \ddot{\theta}^2 - \frac{1}{2} m_2 g l \cos \theta 

\Rightarrow \frac{d}{dt} \frac{m_1 m_2 l^2}{m_1 + m_2} \ddot{\theta} - \left( -m_2 g l \cos \theta \right) = 0 

\Rightarrow \frac{m_1 m_2 l^2}{m_1 + m_2} \ddot{\theta} = m_2 g l \cos \theta 

\Rightarrow \ddot{\theta} = -m_2 g l \cos \theta \quad , \quad \omega^2 = \frac{m_2 g l}{m_1 m_2 l^2} = \frac{g}{m_1} 

W = \sqrt{\frac{(m_1 + m_2) g}{m_1 l}} 

(f) \text{ Limit } m_1 \to \infty \Rightarrow \text{ fixed support point } \Rightarrow \text{ ideal pendulum} 

W \Rightarrow \sqrt{\frac{g}{l}} \quad , \text{ as it should be.} 

\text{ Limit } m_1 \to 0 \Rightarrow W \to \infty. \text{ Why? This is free sliding.}
Problem 9 - Electricity and Magnetism/Optics

Consider a monochromatic plane electromagnetic wave incident on a planar interface. The electromagnetic field $\vec{E}$ of this plane wave can be expressed as $\vec{E} = \vec{E}_0 \cos \left( \vec{k} \cdot \vec{r} - \omega t \right)$. The reflected and transmitted waves can be expressed as $\vec{E}_r = \vec{E}_{0r} \cos \left( \vec{k}_r \cdot \vec{r} - \omega t \pm \delta \right)$ and $\vec{E}_t = \vec{E}_{0t} \cos \left( \vec{k}_t \cdot \vec{r} - \omega t + \varepsilon \right)$. Here $\vec{E}_{0r}$, $\vec{E}_{0t}$ and $\vec{k}$ are constant vector amplitudes. Above the interface the index of refraction of that medium is $n_i$, the medium below the interface has an index of refraction $n_f$. The vector $\hat{n}$ is a unit vector normal to the plane of the interface directed downward. Choose the coordinate system such that the plane of the interface lies at constant $y = 0$.

(a) [5 pts.] State the boundary conditions on $\vec{E}$ at the interface boundary. Express this as a vector relation that relates $\vec{n}$, $\vec{E}_i$, $\vec{E}_r$ and $\vec{E}_t$.

(b) [5 pts.] From the vector relation obtained for the boundary condition and the fact that the relation must be satisfied at any instant in time and at any point on the interface $y = 0$, obtain an expression that relates the terms $\left( \vec{k}_i \cdot \vec{r} - \omega t \right)_{y=0}$, $\left( \vec{k}_r \cdot \vec{r} - \omega t + \varepsilon \right)_{y=0}$ and $\left( \vec{k}_t \cdot \vec{r} - \omega t + \delta \right)_{y=0}$. The frequencies of the transmitted and reflected wave must equal the frequency of the incident wave $\omega = \omega_i = \omega_r$. Show that the angle of reflection must equal the angle of incidence $\theta_i = \theta_r$.

(c) [5 pts.] From the boundary condition, derive Snell’s Law $n_i \sin \theta_i = n_f \sin \theta_f$.

(d) [10 pts.] For the case that the electric field is perpendicular to the plane of incidence, calculate the ratio of the amplitudes of the reflected to the incident electric field $\left( \frac{E_{0r}}{E_{0i}} \right)$ and the ratio of the of the amplitudes of the transmitted to incident electric field $\left( \frac{E_{0t}}{E_{0i}} \right)$. 


Problem 9: E&M/Optics

See textbook "Optics" by Hecht & Zajac pp 71-75

\( \hat{u}_n \) is the downward pointing unit vector normal to the interface.

a) Boundary condition on \( \vec{E} \) at the interface boundary \( \hat{t} \):

The component of \( \vec{E} \) that is tangent to the interface must be continuous at the interface.

At the interface:

\[
\begin{align*}
(\vec{E}_\parallel)_r &= \hat{u}_n \times \vec{E}_t^r \\
(\vec{E}_\perp)_r &= \hat{u}_n \times \vec{E}_t^r \\
(\vec{E}_t)_\parallel &= \hat{u}_n \times \vec{E}_t^r
\end{align*}
\]  

Boundary condition \( \Rightarrow \)

\[
(\vec{E}_\parallel)_r + (\vec{E}_\perp)_r = (\vec{E}_t)_\parallel
\]  

Expressed in terms of \( \hat{u}_n, \vec{E}_t^r, \vec{E}_t^r, \) and \( \vec{E}_t^r \):

\[
\hat{u}_n \times \vec{E}_t^r + \hat{u}_n \times \vec{E}_t^r = \hat{u}_n \times \vec{E}_t^r
\]
9) Expressing boundary condition fully

\[ \hat{n} \times \mathbf{E}_0 \cos (k_z \cdot \vec{r} - \omega_z t) + \hat{n} \times \mathbf{E}_r \cos (k_r \cdot \vec{r} - \omega_r t + \epsilon_r) = \hat{n} \times \mathbf{E}_0 \cos (k_z \cdot \vec{r} - \omega_z t + \epsilon_z) \]  \hspace{1cm} \hspace{1cm} (6)

where \( \vec{r} \) is the vector from the origin to the point on the interface under consideration.

The relationship expressed in equation 6 above holds for all values of \( t \) & \( \vec{r} \) on the interface. Consequently \( E_z, E_r \) & \( E_t \) must have the same functional dependence on the variables \( \vec{r} \) & \( t \)

\[ \Rightarrow \left( \frac{\partial k_z \cdot \vec{r} - \omega_z t}{\partial y} \right)_{y=0} = \left( \frac{\partial k_r \cdot \vec{r} - \omega_r t + \epsilon_r}{\partial y} \right)_{y=0} \]

\[ = \left( \frac{\partial k_t \cdot \vec{r} - \omega_t t + \epsilon_t}{\partial y} \right)_{y=0} \]  \hspace{1cm} \hspace{1cm} (7)

Furthermore for this equality to hold for all values of \( t \) \( \Rightarrow \omega_z = \omega_t = \omega_r \).

One can then cancel the terms dependent on time to obtain

\[ \left( \frac{\partial \vec{r}}{\partial y} \right)_{y=0} = \left( \frac{\partial \vec{r}}{\partial y} \right)_{y=0} = \left( \frac{\partial k_t \cdot \vec{r} + \epsilon_t}{\partial y} \right)_{y=0} \]  \hspace{1cm} \hspace{1cm} (8)

From the first two terms we obtain

\[ \left[ \left( \frac{\partial \vec{r}}{\partial y} \right) \times \vec{r} \right]_{y=0} = \mathbf{E}_r \] \hspace{1cm} \hspace{1cm} (9)

The value of the phase \( \phi_r \) depends on the choice of the origin. The above expression is for a plane where the...
(9) b) endpoint of $\mathbf{r}$ is perpendicular to $(\mathbf{R}_i - \mathbf{R}_r)$. This plane is also the plane of the interface. Therefore $(\mathbf{R}_i - \mathbf{R}_r)$ is perpendicular to the interface & parallel to $\mathbf{u}_n$

$$\Rightarrow \quad \mathbf{u}_n \times (\mathbf{R}_i - \mathbf{R}_r) = 0 \quad (10)$$

$$= \quad k_i \sin \theta_i = k_r \sin \theta_r \quad (11)$$

Since $n_i = n_r \Rightarrow k_i = k_r$

$$\therefore \quad (\theta_i = \theta_r) \quad (12)$$

(9c) From eqn (7) we can also obtain

$$\left[ (\mathbf{R}_i - \mathbf{R}_t) \cdot \mathbf{n} \right] \bigg|_{y=0} = \varepsilon_t \quad (13)$$

$$\Rightarrow \quad \left[ \mathbf{R}_i - \mathbf{R}_t \right] \text{ is normal to the interface. The tangential components of } \mathbf{R}_i \text{ & } \mathbf{R}_t \text{ must be equal}$$

$$\Rightarrow \quad k_i \sin \theta_i = k_t \sin \theta_t \quad (14)$$

using $k = n \frac{w}{c}$ & $w_i = w_t$ we obtain Snell's law

$$n_i \sin \theta_i = n_t \sin \theta_t \quad (15)$$
(9) d) If the coordinate origin $O$ is chosen to be in the interface ($y=0$, $z_r = z_t = 0$)

For $E_\perp$ plane of incidence

\[ \vec{B} \parallel \text{plane of incidence} \]

\[ \vec{\nabla} \times \vec{E} = \nu \vec{B} \]

\[ \vec{\nabla} \cdot \vec{E} = 0 \] (116) (117)

Tangential $E$ continuous from Boundary conditions

\[ \Rightarrow \vec{E}_{0i} + \vec{E}_{0r} = \vec{E}_{0t} \quad \text{@ interface} \] (118)

Boundary conditions

\[ \varepsilon \vec{E}_\perp \text{ continuous across interface} \]

\[ \vec{B}_\perp \text{ continuous across interface} \]

\[ \vec{B}_\parallel /\mu \text{ continuous across interface} \]

\[ \vec{E}_\parallel \text{ continuous across interface} \]

Continuity of tangential $\vec{B}/\mu$

\[ \Rightarrow -\frac{\vec{B}_r}{\mu} \cos \theta_i + \frac{\vec{B}_r}{\mu} \cos \theta_r = -\frac{\vec{B}_r}{\mu} \cos \theta_t \] (119)
From $\vec{J}_2 \times \vec{E} = \sqrt{B}

B_i = E_i / v_i
B_r = E_r / v_r
B_t = E_t / v_t

Since $v_i = v_r \cdot \Theta_i = \Theta_r$ eqn 19 can be written as

$$\frac{1}{m_\text{i} v_\text{i}} (E_\text{i} - E_\text{r}) \cos \Theta_\text{i} = \frac{1}{m_\text{t} v_\text{t}} E_\text{t} \cos \Theta_\text{t} \quad (20)$$

Using definitions of $E_\text{i}, E_\text{r}, E_\text{t}$ and that the cosines of said functions are all equal to $y = 0$ one obtains from (19)

$$\frac{n_\text{i}}{M_\text{i}} (E_\text{r} - E_\text{r}) \cos \Theta_\text{i} = \frac{n_\text{t}}{M_\text{t}} E_\text{t} \cos \Theta_\text{t}$$

Combining with $E_\text{r} + E_\text{r} = E_\text{t}$

$$\Rightarrow \left( \begin{array}{c} E_\text{r} \\ E_\text{r} \end{array} \right) = \left( \begin{array}{c} \frac{n_\text{i}}{M_\text{i}} \cos \Theta_\text{i} - \frac{n_\text{t}}{M_\text{t}} \cos \Theta_\text{t} \\ \frac{n_\text{i}}{M_\text{i}} \cos \Theta_\text{i} + \frac{n_\text{t}}{M_\text{t}} \cos \Theta_\text{t} \end{array} \right)$$

and

$$\left( \begin{array}{c} E_\text{t} \\ E_\text{r} \end{array} \right) = \left( \begin{array}{c} \frac{2 n_\text{i}}{M_\text{i}} \cos \Theta_\text{i} \\ \frac{n_\text{i}}{M_\text{i}} \cos \Theta_\text{i} + \frac{n_\text{t}}{M_\text{t}} \cos \Theta_\text{t} \end{array} \right)$$
Problem 10 - Quantum Mechanics

Consider a hydrogen atom that is placed in an external magnetic field $B$ which is very strong compared to the internal magnetic field of the hydrogen atom itself. The orbital and spin magnetic moments will then interact independently with the external magnetic field. The energy levels of the Hydrogen atom in the external magnetic filed will depend on both the orbital and spin quantum numbers $m_l$ and $m_s$.

(a) [5 pts.] Evaluate the splitting of the energy levels according to the values of $m_l$ and $m_s$.

(b) [5 pts.] Draw the pattern of split levels originating from the $n=2$ level, enumerating the quantum numbers of each component of the pattern.

(c) [10 pts.] Calculate the strength of the external magnetic field that would produce an energy difference between the most widely separated $n=2$ levels that would equal the difference between the $n = 1$ and $n = 2$ levels in the absence of the external magnetic field.

(d) [5 pts.] What determines the selection rules for transitions between atomic energy levels via the emission or absorption of a single photon? Enumerate the allowed transitions from the $n=2$ to $n=1$ levels.
Problem 10
QM solution

a) SPLITTING OF ENERGY LEVELS

\[ \Delta E = \frac{\epsilon h}{2m_e} B \left( m_x + 5m_y \right) \]

b)\[
\begin{array}{c}
\hline
n=2, \quad l=1 \\
\hline
m_x = 1, \quad m_y = \pm \frac{1}{2} \\
m_x = 0, \quad m_y = \pm \frac{1}{2} \\
m_x = -1, \quad m_y = \pm \frac{1}{2} \\
\hline
\end{array}
\]
\[ \Phi = 0 \]

Energy difference of Hydrogen \( n=2 \) & \( n=1 \) levels with no B field

\[ |E_2 - E_1| = 13.6 \text{ eV} \left(1 - \frac{1}{2^2}\right) = 10.2 \text{ eV} \]

- \( \Delta E \) due to B-field splittings

\[ \Delta E = 2 \mu_B B \left( 1 + 2 \cdot \frac{1}{2} \right) = 4 \mu_B B = 10.2 \text{ eV} \]

\[ B = \frac{10.2 \text{ eV}}{4 \mu_B} = \frac{10.2 \text{ eV}}{4.57 \times 10^{-5} \text{ eV/T}} = 4.4 \times 10^4 \text{ Tesla} \]

which is a phenomenally large field attainable only by objects such as neutron stars.

d) SELECTION RULES FOR ATOMIC TRANSITIONS DETERMINED BY THE FACT THAT PHOTON CARRIES AWAY 1 \( \hbar \) WORTH OF ANGULAR MOMENTUM.
photon $S=1, m_S = \pm 1$ 0 is not allowed due to S.R.

$\Rightarrow \quad \Delta \ell = \pm 1$

$\Delta m_\rho = 0 \quad or \quad \pm 1$

$\Delta m_\gamma = 0$

total angular momentum $\Delta j = 0 \quad or \quad \pm 1$

$\Delta t = \pm \frac{n}{2} m_e$