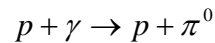


Common Exam Prep Final Review Problem **Aug. 15, 2007**

The Cosmic Microwave Background Radiation (CMBR) refers to the *ideal* gas of photons that permeates the Universe, whose temperature is measured to be $T=2.7$ K.

- (a) Can a gas consisting purely of photons and nothing else have a temperature? Explain.
- (b) Remembering that photons are mass-less, find an expression for the (normalized) probability density $p(E)$ that a photon has energy between E and $E+dE$.
- (c) Find the energy, E_S , in electron-volts (eV), which marks the +1 standard deviation point of the energy distribution. In other words, only 16% of the photons have energies above E_S . Carry the calculation out symbolically as far as you can before plugging in numbers.
- (d) Assuming a photon of this energy (E_S) collides *head-on* with a cosmic-ray proton of energy E . Find the value of E , in eV, above which it is possible for the following reaction to occur:



The mass energy of a proton is $m_p c^2=936$ MeV and that of a pion is $m_\pi c^2=140$ MeV. Again, carry the calculation out symbolically as far as you can before plugging in numbers. (This is rather a crude estimate of what is known as the Greisen-Zatsepin-K'uzmin Cut-Off, recently discovered by the HiRes group hosted by University of Utah).

(a) Technically, photons do NOT interact with one another, so that they have no mechanism to exchange energy (i.e. photon-photon collisions are not allowed in the leading order). In this way they cannot come to equilibrium, which is required for "temperature" to be meaningful.

The fact that the CMBR appears to be at equilibrium means that they are at equilibrium with the inter-stellar matter, so that the 2.7°K temperature is also that of the interstellar medium.

(b) photons have states ~~specified~~ specified by $p_x, p_y, p_z, (x, y, z)$

↑
if treated as classical.

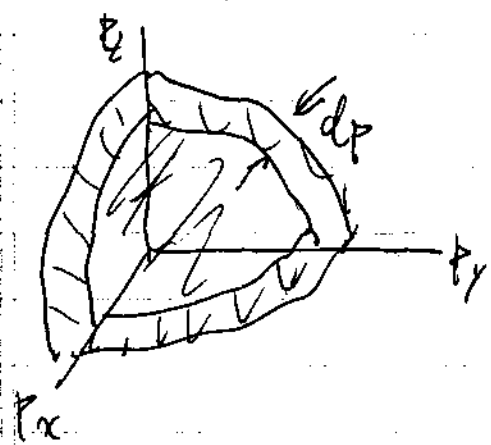
but since we have no "potential" energy, it does not matter

$$\text{For photons: } E^2 = (p_x^2 + p_y^2 + p_z^2)c^2 + \cancel{m^2 c^4}$$

$$\Rightarrow E = pc \quad \text{where } p = |\vec{p}|$$

(b) cont'd

⇒ # of states between p and dp is then given by



$dN = A dV_p$ (momentum phase space volume)
 ↑
 arbitrary normalization constant

$$dV_p = \frac{\pi}{2} p^2 dp$$

↑
($\frac{1}{8} \times 4\pi$)

of states between $E \rightarrow E + dE$

$$dN = \frac{A\pi}{2} p^2 dp$$

but $p = \frac{E}{c} \Rightarrow dp = \frac{dE}{c}$

$$\Rightarrow dN = \frac{A\pi E^2}{2c^3} dE$$

by defn $dN = P(E)dE$ $g(E) =$ "density" of states

$$\Rightarrow g(E) = \frac{A \cdot \pi E^2}{2c^3}$$

$$P(E) = g(E) e^{-E/kT}$$

$$\Rightarrow P(E) = \frac{A\pi E^2}{2c^3} e^{-E/kT}$$

where "A" is chosen to normalize the integral

(c) By defn:

$$P(E' < E) = \int_0^E P(E') dE'$$

and proper normalization demands

that

$$P(\infty) = \int_0^{\infty} P(E') dE' = 1$$

$$\Rightarrow \frac{A\pi}{2c^3} \int_0^{\infty} E'^2 e^{-E'/kT} dE' = 1$$

$$\Rightarrow A = \frac{1}{\frac{\pi}{2c^3} \int_0^{\infty} E'^2 e^{-E'/kT} dE'}$$

$$\Rightarrow P(E' < E) = \frac{\frac{\pi}{2c^3} \int_0^E E'^2 e^{-E'/kT} dE'}{\frac{\pi}{2c^3} \int_0^{\infty} E'^2 e^{-E'/kT} dE'}$$

$$\int_0^E E'^2 e^{-E'/kT} dE'$$

$$= -kT \left[e^{-E'/kT} (E'^2 + 2E'kT + 2k^2T^2) - 2k^2T^2 \right]$$

$$\int_0^{\infty} E'^2 e^{-E'/kT} dE' = 2k^3T^3$$

(c) cont'd

$$\Rightarrow P(E' < E) = \frac{-1}{2k^2T^2} \left[e^{-E/kT} (E^2 + 2EkT + 2k^2T^2) - 2k^2T^2 \right]$$
$$= \frac{-1}{2} \beta^2 \left[e^{-\beta E} \left(E^2 + \frac{2E}{\beta} + \frac{2}{\beta^2} \right) - 2 \frac{1}{\beta^2} \right]$$

where $\beta = \frac{1}{kT}$

$$\text{and } P(E' < E) = -\frac{e^{-\beta E}}{2} (\beta^2 E^2 + 2\beta E + 2) + 1$$
$$= 1 - \frac{1}{2} e^{-\beta E} (\beta^2 E^2 + 2\beta E + 2)$$

Use MAPLE to solve for $P(E' < E) = 0.84$

let $x = \beta E$

$$\Rightarrow 1 - \frac{1}{2} e^{-x} (x^2 + 2x + 2) = 0.84$$

$$\Rightarrow e^{-x} (x^2 + 2x + 2) = 0.32$$

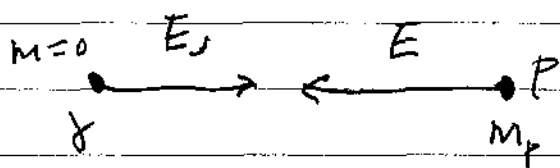
graphically $\rightarrow x \approx 4.6$

$$\text{i.e.: } E_x \sim 4.6 kT = (4.6) \left(1.38 \times 10^{-23} \frac{\text{J}}{\text{K}} \right) (2.7 \text{K})$$

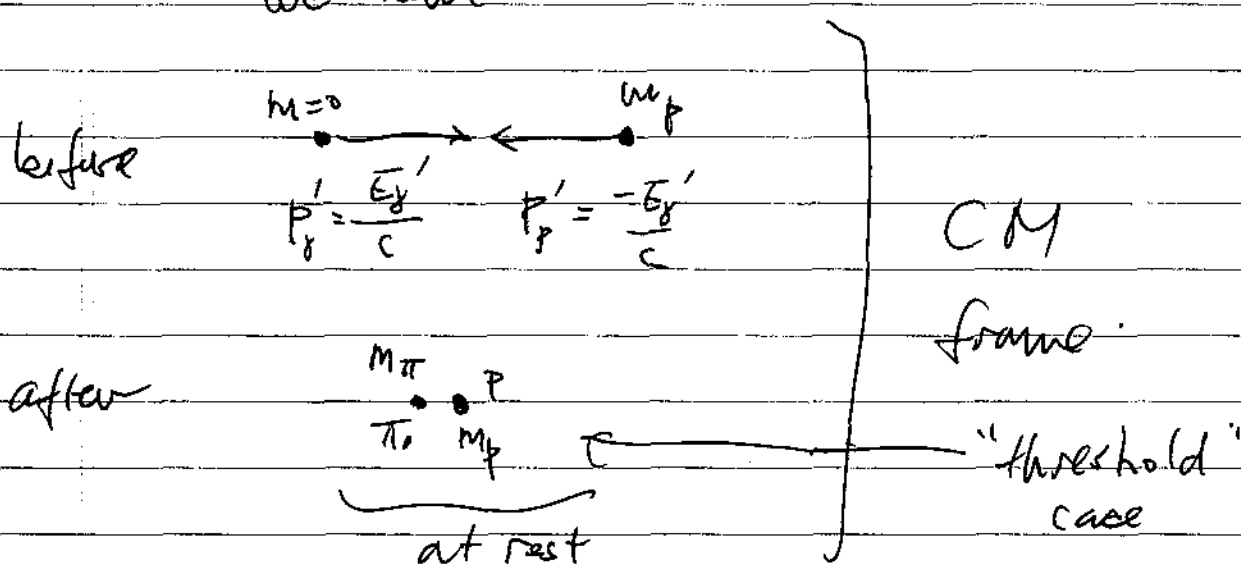
$$= 1.71 \times 10^{-22} \text{ J} = 0.00107 \text{ eV}$$

$$\approx 1 \text{ meV}$$

(d) The collision:



In the "CM" frame: (total momentum = 0)
we have



Invariant mass (magnitude of 4-vector P)
is conserved!
AND Lorentz-invariant

In lab: before collision

$$P_{\text{before}} = \begin{bmatrix} E_s/c \\ E_s/c \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} E/c \\ -\frac{1}{c}(E^2 - m_p^2 c^4)^{1/2} \\ 0 \\ 0 \end{bmatrix}$$

γ P

$$\Rightarrow p_{\text{before}}^2 = \frac{1}{c^2} E_S E + \frac{1}{c^2} E_S (E^2 - m_p^2 c^4)^{\frac{1}{2}}$$

C.M. frame: after collision:

$$p'_{\text{after}} = \begin{bmatrix} m_{\pi} c^2 / c \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} m_p c^2 / c \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (m_{\pi} + m_p) c^2 / c \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{let } m_{\pi} + m_p \equiv M$$

$$\Rightarrow p'_{\text{after}}{}^2 = M^2 c^2$$

$$p_{\text{before}}^2 = p'_{\text{after}}{}^2$$

$$\Rightarrow \frac{1}{c^2} \left[E_S E + E_S (E^2 - m_p^2 c^4)^{\frac{1}{2}} \right] = M^2 c^2$$

$$\Rightarrow E_S \left[E + (E^2 - m_p^2 c^4)^{\frac{1}{2}} \right] = M^2 c^4$$

$$\Rightarrow E + (E^2 - m_p^2 c^4)^{\frac{1}{2}} = \frac{M^2 c^4}{E_S}$$

$$\cancel{E}^2 - m_p^2 c^4 = \left(\frac{M^2 c^4}{E_S} \right)^2 - 2 \frac{M^2 c^4 E}{E_S} + \cancel{E}^2$$

$$\Rightarrow 2E \left(\frac{M^2 c^4}{E_s} \right) = \left(\frac{M^2 c^4}{E_s} \right)^2 - m_p^2 c^4$$

$$2E = \frac{M^2 c^4}{E_s} - \frac{m_p^2 c^4 E_s}{M^2 c^4}$$

$$E = \frac{1}{2} \left[\frac{E_M^2}{E_s} - \frac{E_p^2 E_s}{E_M^2} \right]$$

$$E_p = m_p c^2 = 936 \times 10^6 \text{ eV} = 9.36 \times 10^8 \text{ eV}$$

$$M c^2 = E_M = (936 + 140) \times 10^6 \text{ eV} = 1.076 \times 10^9 \text{ eV}$$

$$E_s = 1.07 \times 10^{-3} \text{ eV}$$

$$E = \frac{1}{2} \left[\frac{(1.076 \times 10^9)^2}{1.07 \times 10^{-3}} - \frac{(9.36 \times 10^8)^2 (1.07 \times 10^{-3})}{(1.076 \times 10^9)^2} \right] \text{ eV}$$

↑
2nd term is very small
compared to the 1st

$$E \approx 5 \times 10^{20} \text{ eV}$$

```
> assume(k>0, T>0);
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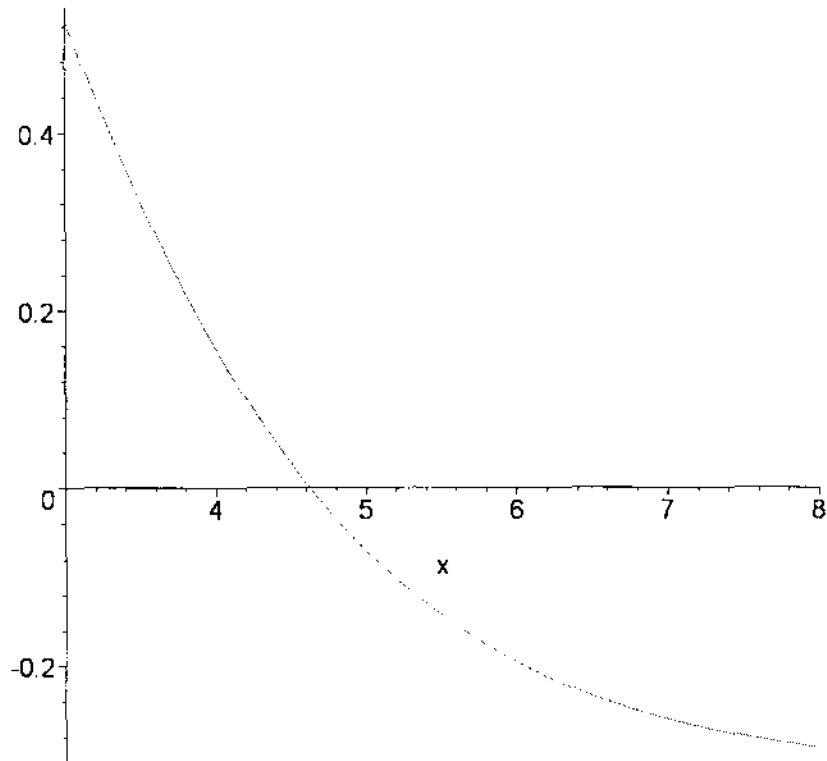
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> simplify(int(u^2*exp(-u/k/T), u=0..E));
```

$$-kT \left(E^2 e^{\left(-\frac{E}{kT}\right)} + 2E e^{\left(-\frac{E}{kT}\right)} kT + 2 e^{\left(-\frac{E}{kT}\right)} k^2 T^2 - 2k^2 T^2 \right)$$

```
> f(x) := exp(-x) * (x^2 + 2*x + 2) - 0.32;
```

$$f(x) := e^{-x} (x^2 + 2x + 2) - 0.32$$

```
> plot(f(x), x=3..8);
```



```
>
```