A. Single-layer graphene is a two-dimensional gapless semiconductor with mirror symmetry of conduction and valence bands (Fig. 1). The dispersion law, which is the dependence of electron energy, $\varepsilon$, on momentum, $\vec{p}$, is linear, $\varepsilon = |\vec{p}| v$. At zero temperature, $T = 0$, the conduction band is empty, while the valence band is completely occupied.

1. With accuracy of a numeric coefficient find the concentrations of electrons and holes at finite temperature $T$.
2. Find the average electron energy and the specific heat.

B. In typical three dimensional gapless semiconductor the dispersion laws for electrons and holes are quadratic:

$$\varepsilon_e = \frac{p^2}{2m_e}, \quad \varepsilon_h = -\frac{p^2}{2m_h}.$$ 

The hole mass is much bigger than the electron mass $m_h \gg m_e$. Using this fact, perform the same calculations 1 and 2 as in part A.
Solution Part I

Density of electron states
\[ g_e(\varepsilon) = 2 \int \frac{d^2p}{(2\pi \hbar)^2} \delta(\varepsilon - \varepsilon(p)) = \frac{\varepsilon}{\pi \hbar^2} \]

Concentrations of electrons and holes are equal. Due to the symmetry of conduction and valence bands, the Fermi level is zero, \( \mu = 0 \), at any \( T \).

Electron concentration \[ n_e = \int d\varepsilon \frac{g_e(\varepsilon)}{e^{\varepsilon/T} + 1} = \frac{T^2}{\pi(\hbar T)^2} \int_0^\infty \frac{dz}{e^{z/T} + 1} \]

Thus, \( n_e \propto T^2 \)

The average electron energy \[ \bar{E} = \int_0^\infty d\varepsilon \frac{\varepsilon g_e(\varepsilon)}{e^{\varepsilon/T} + 1} = \frac{T^3}{\pi(\hbar T)^2} \int_0^\infty \frac{dz z^2}{e^{z/T} + 1} \]

Thus, the specific heat is proportional to \( T^2 \).

Part II \[ g_e(\varepsilon) = 2 \int \frac{d^2p}{(2\pi \hbar)^2} \delta(\varepsilon - \varepsilon(p)) = \frac{(2m_e)^{3/2}}{2\pi^2 \hbar^3} \sqrt{\varepsilon} \]

The density of the hole states is determined from the electroneutrality condition
\[ \int_0^\infty d\varepsilon \frac{g_e(\varepsilon)}{e^{\varepsilon/T} + 1} = \int_0^\infty d\varepsilon \frac{g_h(\varepsilon)}{e^{\varepsilon/\mu T} + 1} \]

Thus \( \mu \gg T \), so that the left-hand-side is \( \frac{2}{3} \mu^{3/2} \), while the right-hand-side is \( (m_e h T)^{3/2} \sqrt{\pi} e^{\mu/\mu T} \). Therefore \( \mu \approx \frac{2}{3} T \ln\left(\frac{m_e h T}{m_e}\right) \) and \( n_e \propto T^{3/2} \)