The ground state of hydrogen can be described by an electron wavefunction of the form $\psi_{1s}(r) = A \exp(-r/a_0)$.

(a) Compute $A$.

(b) Where is the electron most likely to be? Derive the most probable value of the radial coordinate $r$ of the electron wavefunction.

(c) Can you guess what the “expectation value” of the $x$-component of the momentum of the electron in the Hydrogen atom is? There is a very simple argument.

(d) Show that $\psi_{1s}$ is a solution of the 3D Schrödinger equation.

(e) An unknown magnetic field produces a set of spectral lines for a transition from a state with $l = 3$ to one with $l = 2$ in hydrogen. The maximum energy difference between these lines (i.e. the spread in energy induced by the magnetic field) is 0.61 meV. How many lines are observed in total? What is the magnitude of the magnetic field?
Given: Schrödinger equation in 3D

\[-\frac{\hbar^2}{2M} \nabla^2 \psi + V(r)\psi = E \psi (r, \theta, \phi)\]

\[-\frac{\hbar^2}{2M} \left[ \frac{\partial^2}{r^2 \partial r^2} (r^2 \frac{\partial}{\partial r}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (r^2 \sin \theta \frac{\partial}{\partial \theta}) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] \psi (r, \theta, \phi) + V(r)\psi = E \psi\]

\[d(\text{vol}) = dr (rd\theta)(r \sin \theta d\phi) \text{ volume element}\]

Gamma functions

\[\Gamma(-3/2) = \frac{4\sqrt{\pi}}{3} \approx 2.363\]
\[\Gamma(-1/2) = -2\sqrt{\pi} \approx -3.545\]
\[\Gamma(1/2) = \sqrt{\pi} \approx 1.772\]
\[\Gamma(1) = 0! = 1\]
\[\Gamma(3/2) = \frac{\sqrt{\pi}}{2} \approx 0.886\]
\[\Gamma(2) = 1! = 1\]
\[\Gamma(5/2) = \frac{3\sqrt{\pi}}{4} \approx 1.329\]
\[\Gamma(3) = 2! = 2\]
\[\Gamma(7/2) = \frac{15\sqrt{\pi}}{8} \approx 3.323\]
\[\Gamma(4) = 3! = 6\]

Definite integrals lacking closed-form antiderivatives

There are some functions whose antiderivatives cannot be expressed in closed form. However, the values of the definite integrals of some of these functions over some common intervals can be calculated. A few useful integrals are given below:

\[\int_0^\infty \sqrt{x} e^{-x} \, dx = \frac{1}{2}\sqrt{\pi} \text{ (see also Gamma function)}\]
\[\int_0^\infty e^{-x^2} \, dx = \frac{1}{2}\sqrt{\pi} \text{ (the Gaussian integral)}\]
\[\int_0^\infty x^n e^{-x} \, dx = \frac{n!}{e} \text{ (see also Binomial coefficient)}\]
\[\int_0^\infty x^n e^{-x} \, dx = \frac{n!}{e} \text{ (see also Gamma function)}\]
\[\int_0^\frac{\pi}{2} \sin^2 x \, dx = \frac{\pi}{2}\]
\[\int_0^\frac{\pi}{2} \cos^2 x \, dx = \frac{\pi}{2}\]
\[\int_0^\frac{\pi}{2} \sin^3 x \, dx = \frac{3}{2}\]
\[\int_0^\frac{\pi}{2} \cos^3 x \, dx = \frac{3}{2}\]
\[\int_0^\infty x^{n-1} e^{-x} \, dx = \Gamma(n) \text{ (where } \Gamma(n) \text{ is the Gamma function)}\]
\[\int_0^\infty e^{-(x^2+a^2)} \, dx = \sqrt{\frac{\pi}{a}} \exp \left[ \frac{\beta^2 - 4ac}{4a} \right] \text{ (where } \exp(x) \text{ is the exponential function } e^x)\]
\[\int_0^{2\pi} e^{i\alpha \cos \theta} \, d\theta = 2\pi I_0(a) \text{ (where } I_0(a) \text{ is the modified Bessel function of the first kind)}\]
\[\int_0^{2\pi} e^{i\alpha \cos \theta + i\beta \sin \theta} \, d\theta = 2\pi I_0 \left( \sqrt{a^2 + \beta^2} \right)\]
\[\int_{-\infty}^{\infty} \left( 1 + \frac{x^2}{\nu} \right)^{-\nu/2} \, dx = \frac{\sqrt{\pi \nu}}{\Gamma(\nu/2)} \Gamma(\nu + 1/2) \text{ (this is related to the probability density function of the Student's t-distribution)}\]