Common Exam Prep Question
July 18, 2007

The ground state of hydrogen can be described by an electron wavefunction of the form $_{100}(r) = A\ \exp(-r/a_0)$.

(a) Compute A.

(b) Where is the electron most likely to be? Derive the *most probable value* of the radial coordinate r of the electron wavefunction.

(c) Can you guess what the "expectation value" of the x-component of the momentum of the electron in the Hydrogen atom is? There is a very simple argument.

(d) Show that $_{100}$ is a solution of the 3D Schrödinger equation.

(e) An unknown magnetic field produces a set of spectral lines for a transition from a state with $l = 3$ to one with $l = 2$ in hydrogen. The maximum energy difference between these lines (i.e. the spread in energy induced by the magnetic field) is 0.61 meV. How many lines are observed in total? What is the magnitude of the magnetic field?

Given: Schrödinger equation in 3D $\frac{-\hbar^2}{2M}\nabla^2\psi + V(r)\psi = E\psi(r,\theta,\phi)$ $\frac{-\hbar^2}{2M}[\frac{\partial}{r^2\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}]\psi(r,\theta,\phi) + V(r)\psi = E\psi$ $d(vol) = dr(rd\theta)(r\sin\theta d\phi)$ volume element	$\Gamma(-3/2) = \frac{4\sqrt{\pi}}{3} \approx 2.363$ $\Gamma(-1/2) = -2\sqrt{\pi} \approx -3.545$ $\Gamma(1/2) = \sqrt{\pi} \approx 1.772$ $\Gamma(1) = 0! = 1$ $\Gamma(3/2) = \frac{\sqrt{\pi}}{2} \approx 0.886$ $\Gamma(2) = 1! = 1$ $\Gamma(5/2) = \frac{3\sqrt{\pi}}{4} \approx 1.329$ $\Gamma(3) = 2! = 2$ $\Gamma(7/2) = \frac{15\sqrt{\pi}}{8} \approx 3.323$ $\Gamma(4) = 3! = 6$ Gamma functions

Definite integrals lacking closed-form antiderivatives

[edit]

There are some functions whose antiderivatives *cannot* be expressed in closed form. However, the values of the definite integrals of some of these functions over some common intervals can be calculated. A few useful integrals are given below.

$$\int_0^\infty \sqrt{x}\,e^{-x}\,dx = \frac{1}{2}\sqrt{\pi}$$ (see also Gamma function)

$$\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}$$ (the Gaussian integral)

$$\int_0^\infty \frac{x}{e^x-1}\,dx = \frac{\pi^2}{6}$$ (see also Bernoulli number)

$$\int_0^\infty \frac{x^3}{e^x-1}\,dx = \frac{\pi^4}{15}$$

$$\int_0^\infty \frac{\sin(x)}{x}\,dx = \frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx = \frac{1\cdot3\cdot5\cdot\cdots\cdot(n-1)}{2\cdot4\cdot6\cdot\cdots\cdot n}\frac{\pi}{2}$$ (if n is an even integer and $n \geq 2$)

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx = \frac{2\cdot4\cdot6\cdot\cdots\cdot(n-1)}{3\cdot5\cdot7\cdot\cdots\cdot n}$$ (if n is an odd integer and $n \geq 3$)

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$$

$$\int_0^\infty x^{z-1}\,e^{-x}\,dx = \Gamma(z)$$ (where $\Gamma(z)$ is the Gamma function)

$$\int_{-\infty}^\infty e^{-(ax^2+bx+c)}\,dx = \sqrt{\frac{\pi}{a}}\exp\left[\frac{b^2-4ac}{4a}\right]$$ (where $\exp[u]$ is the exponential function e^u.)

$$\int_0^{2\pi} e^{x\cos\theta}d\theta = 2\pi I_0(x)$$ (where $I_0(x)$ is the modified Bessel function of the first kind)

$$\int_0^{2\pi} e^{x\cos\theta+y\sin\theta}d\theta = 2\pi I_0\left(\sqrt{x^2+y^2}\right)$$

$$\int_{-\infty}^\infty (1+x^2/\nu)^{-(\nu+1)/2}dx = \frac{\sqrt{\nu\pi}\ \Gamma(\nu/2)}{\Gamma((\nu+1)/2))}$$ ($\nu > 0$, this is related to the probability density function of the Student's t-distribution)