Common Exam Prep Question
June 20, 2007

A. Consider electromagnetic waves propagating between two metallic planes, $x = d/2$ and $x = -d/2$ (waveguide modes, Fig. 1).

i. Find the wavevectors of the modes along the direction of propagation, z, as a function of frequency, ω (dispersion relation).

ii. Find the phase velocities, v_{ph}, and the group velocities, v_g, of the modes.

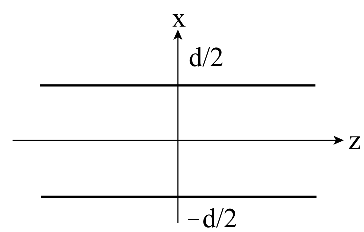

Fig. 1

B. Assume that the distance between the planes drops from d to $d_1 < d$ at $z = \pm D/2$ (Fig. 2).

i. How many waveguide modes will be trapped on the interval $-D/2 < z < D/2$?

ii. Find the frequencies of the trapped modes.

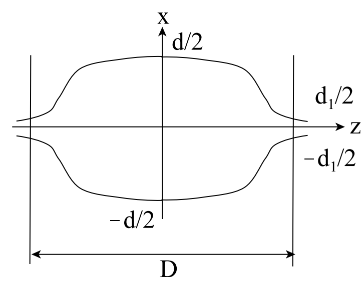

Fig. 2

Solution

I. TE polarization: nonzero component of electric field is E_y. It satisfies the wave equation

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial z^2} + \frac{\omega^2}{c^2} E_y = 0$$ and the boundary conditions

$$E_y(d/2) = E_y(-\tfrac{d}{2}) = 0$$

Solution: $E_y = e^{iqz} \cos(Kx + \varphi)$ with $K^2 + q^2 = \frac{\omega^2}{c^2}$

From the boundary conditions we find $Kd = \pi n$, where $n = 1, 2, 3..$ Thus $q_n = \sqrt{\frac{\omega^2}{c^2} - \frac{\pi^2 n^2}{d^2}}$

$$v_{ph} = \frac{\omega}{q_n} = c \frac{1}{\sqrt{1 - \left(\frac{\pi n c}{\omega d}\right)^2}} \qquad v_{gr} = \frac{\partial \omega}{\partial q} = \frac{1}{\left(\frac{\partial q}{\partial \omega}\right)} = c\sqrt{1 - \left(\frac{\pi n c}{\omega d}\right)^2}$$

TM polarization: E_z satisfies the wave equation and the boundary conditions $E_z(d/2) = E_z(-\tfrac{d}{2}) = 0$

Thus, the dispersion law for TM modes is the same

II The frequencies of the trapped modes are found from the condition $q_N D = \sqrt{\frac{\omega^2}{c^2} - \frac{\pi^2 n^2}{d^2}}\, D = \pi N$, where $N = 1, 2 \ldots$ Thus

$$\omega_{n,N} = \pi c \left[\frac{N^2}{D^2} + \frac{n^2}{d^2}\right]^{1/2}$$

$$\frac{\omega_{n,N}}{c} = \pi \left[\frac{N^2}{D^2} + \frac{n^2}{d^2}\right]^{1/2}$$

The condition of trapping is $< \frac{\pi n}{d_1}$

Therefore, the number of the trapped modes is

$$N = \frac{nD}{d}\sqrt{\frac{d^2}{d_1^2} - 1}$$