A. Consider an electron in a rectangle with hard walls (Fig. 1). The sides of the rectangle are $a$ and $B$.

(a) Find the energy levels and the corresponding wave functions.

(b) How many levels are there with energies smaller than a given value $E$?

B. Consider the case $\frac{a}{B} = \frac{3}{5}$. Find the two levels that are degenerate.
Solution

I. Since the Schrödinger equation allows the separation of variables, the eigenfunctions have the form $\psi(x,y) = \Phi_n(x) \Phi_n(y)$, where the functions $\Phi_n$ and $\Phi_n$ satisfy the equations

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \Phi_n}{\partial x^2} = E_{n_1} \Phi_n$$

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \Phi_n}{\partial y^2} = E_{n_2} \Phi_n$$

and the boundary conditions $\Phi_{n_1}(a/2) = \Phi_{n_1}(-a/2)$ and $\Phi_{n_2}(b/2) = \Phi_{n_2}(-b/2)$, respectively. Then the energy levels are given by $E = \frac{\hbar^2 \pi^2}{2n_1^2 \alpha^2 + 2n_2^2 \beta^2}$.

Condition $E_{n_1, n_2} < E$ can be cast in the form

$$\frac{n_1^2}{(2mEa)^2} + \frac{n_2^2}{(2mEb)^2} < 1$$

The number of positive integers satisfying this condition is

$$N_E = \frac{\pi}{4} \left( \sqrt{\frac{2mEa}{\hbar^2 \pi^2}} \right) \left( \sqrt{\frac{2mEb}{\hbar^2 \pi^2}} \right) = \frac{mEa b}{2\pi \hbar^2}$$

II. \[ \frac{n_1^2}{a^2} + \frac{9}{25} \frac{n_2^2}{a^2} = \frac{N_1^2}{a^2} + \frac{9}{25} \frac{N_2^2}{a^2} \rightarrow 25(n_1^2 - N_1^2) = 9(N_2^2 - n_2^2) \]

The solution is $n_1 = 5$, $n_2 = 12$, $N_1 = 4$, $N_2 = 13$. 