Common Exam Prep Question
June 27, 2007

A. Consider an electron in a rectangle with hard walls (Fig. 1). The sides of the rectangle are a and B.

(a) Find the energy levels and the corresponding wave functions.

(b) How many levels are there with energies smaller than a given value E?

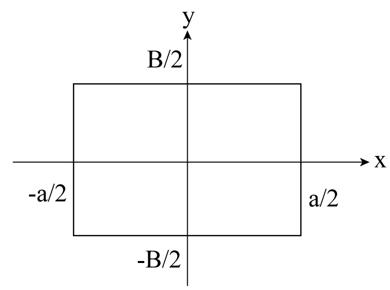

Fig. 1

B. Consider the case $\frac{a}{B} = \frac{3}{5}$. Find the two levels that are degenerate.

Solution

I. Since the Schroedinger equation allows the separation of variables, the eigenfunctions have the form $\psi_{n_1,n_2}(x,y) = \varphi_{n_1}(x)\,\phi_{n_2}(y)$, where the functions φ_{n_1} and ϕ_{n_2} satisfy the equations

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \varphi_{n_1}}{\partial x^2} = E_{n_1}\varphi_{n_1}, \quad -\frac{\hbar^2}{2m}\frac{\partial^2 \phi_{n_2}}{\partial y^2} = E_{n_2}\varphi_{n_2}$$

and the boundary conditions $\varphi_{n_1}(a/2) = \varphi_{n_1}(-\frac{a}{2})$ and $\phi_{n_2}(\frac{b}{2}) = \phi_{n_2}(-\frac{b}{2})$, respectively. Then the energy level positions are given by $\varepsilon_{n_1,n_2} = \frac{\hbar^2\pi^2}{2m}\left(\frac{n_1^2}{a^2}+\frac{n_2^2}{b^2}\right)$

Condition $\varepsilon_{n_1,n_2} < E$ can be cast in the form

$$\frac{n_1^2}{\left(\sqrt{\frac{2mE}{\hbar^2\pi^2}}a\right)^2} + \frac{n_2^2}{\left(\sqrt{\frac{2mE}{\hbar^2\pi^2}}b\right)^2} < 1$$

The number of positive integers satisfying this condition is

$$N_E = \frac{\pi}{4}\left(\sqrt{\frac{2mE}{\hbar^2\pi^2}}a\right)\left(\sqrt{\frac{2mE}{\hbar^2\pi^2}}b\right) = \frac{mEab}{2\pi\hbar^2}$$

II. $$\frac{n_1^2}{a^2} + \frac{9}{25}\frac{n_2^2}{a^2} = \frac{N_1^2}{a^2} + \frac{9}{25}\frac{N_2^2}{a^2} \rightarrow 25(n_1^2 - N_1^2) = 9(N_2^2 - n_2^2)$$

The solution is $n_1 = 5$, $n_2 = 12$, $N_1 = 4$, $N_2 = 13$