Consider the 1-dimensional infinite square well potential described by

\[ V(x) = \begin{cases} 
0 & \text{if } 0 < x < L \\
\infty & \text{otherwise}
\end{cases} \]

1. Write down the time dependent Schrödinger equation for a single particle trapped in this potential well. Separate variables to obtain equations for both the time independent part of the Schrödinger equation and the time dependency. State the spatial boundary conditions for the single particle wavefunction.

2. Solve the time independent equation to find the energy levels and properly normalized wavefunctions for a single particle. Sketch the ground state and first excited state wavefunctions, and the corresponding probabilities.

3. Determine the position expectation value \( \langle x \rangle \) for a particle in the first excited state. *Hint: Is there a coordinate transformation that would make this calculation easier?*

4. At \( t = 0 \), the wave function of the particle is known to be

\[ \psi(x, t = 0) = \sqrt{\frac{2}{13L}} \left\{ 3 \sin \frac{\pi x}{L} + 2 \sin \frac{3\pi x}{L} \right\} \]

What are the possible observed values for the energy of this state? What is the probability of measuring each of these energies?

5. What is the time-dependent wave function for \( t > 0 \), for the above-specified initial state?

6. Suppose three noninteracting spin-0 particles are trapped in this potential well. At \( T = 0 \) K, what is the energy of the last filled state? What is the total energy of the system?

7. Suppose three noninteracting spin-1/2 particles are trapped in this potential well. At \( T = 0 \) K, what is the energy of the last filled state? What is the total energy of the system?
1) \[-\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x,t)}{\partial x^2} + V(x) \psi(x,t) = i \hbar \frac{\partial \psi(x,t)}{\partial t}\]

where \( V(x) = \begin{cases} 0 & \text{if } 0 < x < L \\ \infty & \text{otherwise} \end{cases} \)

Separate variables: let \( \psi(x,t) = X(x) T(t) \),

Substitute into equation above to obtain

\[-\frac{1}{\hbar^2} \frac{1}{2m} \frac{d^2X}{dx^2} + V(x) = i \frac{\hbar}{t} \frac{dT}{dt} = \mathcal{C}\]

Boundary conditions: \( X(0) = X(L) = 0 \)

2) First look at "time" equation for insight into "C".

\[\frac{dT(t)}{dt} + i \frac{\mathcal{C}}{\hbar} T(t) = 0\]

Will have oscillatory solution \( T(t) = e^{-i\mathcal{C}t/\hbar} \),

with angular frequency \( \omega = \mathcal{C}/\hbar \). Thus \( \mathcal{C} \)

is equal to the energy \( E \rightarrow E - \hbar \omega \).

Inside the box, we can write the "space" equation as

\[\frac{d^2X(x)}{dx^2} + \frac{2mE}{\hbar^2} X(x) = 0\]

which will have solutions like \( \cos(kx), \sin(kx) \)

(\( k = \sqrt{\frac{2mE}{\hbar^2}} \)).


\[ \text{ENTITY solutions won't work, since } X(0) = 0. \]

\[ \text{ENTITY solutions work for } X(0) = 0. \]

\[ X(L) = 0 \text{ requires that} \]

\[ kL = \sqrt{\frac{2mE}{h^2}} = n\pi, \quad n = \text{nonzero positive integer} \]

\[ \text{Thus } E_n = \frac{n^2\pi^2h^2}{2mL^2} \]

\[ \Psi_{nl}(x) = A_n \sin \frac{n\pi x}{L} \]

Normalization condition \[ \int_0^L \Psi_n^* \Psi_n \, dx = 1 \]

\[ A_n^2 \int_0^L \sin^2 \left( \frac{n\pi x}{L} \right) \, dx = A_n^2 \cdot \frac{L}{2} \]

\[ \text{Using } \int \sin^2 \gamma \, d\gamma = \gamma - \sin \gamma \cos \gamma \]

\[ : \quad A_n = \sqrt{\frac{2}{L}} \]

\[ \Psi_n(x,t) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} e^{-iE_n t/h} \]

\[ E_n = \frac{n^2\pi^2h^2}{2mL^2} \]

\text{Sketches} 

\[ \rightarrow \]
ψ and ψ*ψ at t = 0 (L = 1)

ψ, n = 1

ψ*ψ, n = 1

ψ, n = 2

ψ*ψ, n = 2
5) \text{ Expectation value } \langle x \rangle = \int x \psi^* \psi \, dx \text{ in general.}

\text{In this case } \langle x \rangle_{\text{a} = 2} = \frac{2}{L} \int_0^L x \sin^2 \frac{2\pi x}{L} \, dx

\text{Note internal! But note that } x \sin^2 x \text{ is an odd function, hence integrates to zero over a symmetric interval.}

\text{This suggests redefining our } x \text{-axis from } 0 \to L \text{ to } -L/2 \to L/2

\begin{array}{|c|c|}
\hline
0 & L \\
\hline
-\frac{L}{2} & \frac{L}{2} \\
\hline
\end{array}

\text{Let } x' = x - \frac{L}{2}

\text{Then } \langle x \rangle_{2} = \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} (x' + \frac{L}{2}) \sin^2 \left( \frac{2\pi x'}{L} + \pi \right) \, dx'

\text{Odd part of } \sin(\gamma x) = -\sin \gamma \text{ integral vanishes}

= \frac{2}{L} \cdot \frac{L}{2} \int_{-\frac{L}{2}}^{\frac{L}{2}} \sin^2 \left( \frac{2\pi x'}{L} \right) \, dx'

\int \sin^2 ax \, dx = \frac{x}{2} - \frac{\sin 2ax}{4a}, \text{ let } a = \frac{2\pi}{L}

\langle x \rangle_{2} = \left[ \frac{x'}{2} - \frac{\sin \left( \frac{4\pi x'}{L} \right)}{4 \cdot 2\pi/L} \right]_{x' = -\frac{L}{2}}^{\frac{L}{2}}

\langle x \rangle_{2} = \frac{L}{2}
4) By inspection, this is a linear combination of the ground (n=1) and 2nd excited (n=3) states, so possible observed energies are

\[ E_1 = \frac{\hbar^2 k^2}{2mL^2} \quad \text{and} \quad E_3 = \frac{9\hbar^2 k^2}{2mL^2} \]

Prove that \( \int \psi^* \psi \, dx \) as a sum of \( \int \psi_1^* \psi_1 \, dx \) and \( \int \psi_3^* \psi_3 \, dx \)

\[
\int \psi^* \psi \, dx = \frac{2}{13L} \int_0^L (3 \sin \frac{2x}{L} + 2 \sin \frac{3x}{L})(3 \sin \frac{2x}{L} + 2 \sin \frac{3x}{L}) \, dx
\]

By orthogonality of sines, cross terms vanish

\[
= \frac{2}{13L} \left[ \int_0^L 9 \sin^2 \frac{2x}{L} \, dx + \int_0^L 4 \sin^2 \frac{3x}{L} \, dx \right]
\]

\[
= \frac{9}{13} \int \psi_1^* \psi_1 \, dx + \frac{4}{13} \int \psi_3^* \psi_3 \, dx
\]

Hence, probability of observing \( E_1 = \frac{9}{13} \)

\( E_3 = \frac{4}{13} \)

5) \( \psi(x,t) = \sqrt{\frac{2}{13L}} \left\{ 3 \sin \frac{2x}{L} e^{-iE_1 t/\hbar} + 2 \sin \frac{3x}{L} e^{-iE_3 t/\hbar} \right\} \)
6) Spin-$\phi$ particles obey Bose-Einstein statistics (are "bosons")
There are no restrictions on the # of bosons in a quantum state, hence at $T=0$ all are at $n=1$.
Energy of last filled state is $E_1$, total energy = $3E_1$.

7) Spin-$\frac{1}{2}$ particles obey Fermi-Dirac statistics (are "fermions")
No two fermions may occupy the same quantum state. However, fermions have an additional quantum number, the "spin," which for spin-$\frac{1}{2}$ particles can be either $+\frac{1}{2}$ (↑) or $-\frac{1}{2}$ (↓).

At $\phi K$, 2 particles will have $n=1$, one ↑ and one ↓.
1 particle will have $n=2$, either spin up or down.
Energy of last filled state is $E_2$.
Total energy of system = $2E_1 + E_2$. 