LECTURE 2: DIFFERENTIAL CIRCUIT PARAMETERS, INPUT AND OUTPUT RESISTANCE, THEVENIN'S THEOREM

Output Resistance

So far we have treated a battery as a pure emf, i.e., the voltage is constant, independent of the current drawn. In fact, it is often the case that the output voltage drops linearly with the current drawn. In such cases, it is an excellent approximation to regard a real, physical battery as consisting of a pure emf in series with a resistor:

![Electrical Circuit Diagram](image)

Often we will consider an emf source where the output is a function of time (a signal generator, if you will). The symbol is:

![Electrical Symbol](image)

Such a device also usually acts like a time-varying pure emf in series with a resistor:

![Electrical Circuit Diagram](image)

R may be called the source resistance, or the output resistance of the source (or the generator).

Now let us go back to the battery in series with a resistor. Suppose we attach a load resistor whose value is $R_L$:

![Electrical Circuit Diagram](image)
What will be the voltage at the terminals of the physical battery? Clearly \( R_S \) and \( R_L \), form a voltage divider, so \( V_{\text{out}} = \frac{V_S R_L}{R_L + R_S} \), and if \( R_L \) should happen to be equal to \( R_S \), then \( V_{\text{out}} = \frac{1}{2} V \). Now you know a simple, but not unique, way to measure output resistance: it is equal to that load which reduces the output voltage by a factor of 2.

You will often encounter the expression "match impedances". Resistance is one type of impedance, so let us consider the significance of matching resistances, which means to use a load resistance equal to the output resistance. The importance of this approach is related to the power dissipated in the load. It is well known that the power dissipated is given by \( P = IV \). (This is a consequence of the definition of \( V \): the energy lost as a charge of one coulomb flows across a potential difference \( V \).) Since in resistors \( V = IR, P = I(IR) = I^2 R = (V/R)^2 R = V^2 / R \). If the output resistance is given by \( R_S \), then the voltage across the load resistor is:

\[
V_L = \frac{V_S R_L}{R_L + R_S}
\]

The power dissipated is \( P_L = \frac{V_L^2}{R_L} = \frac{\left(\frac{V_S R_L}{R_L + R_S}\right)^2}{R_L} = \frac{V_S^2 R_L^2}{(R_L + R_S)^2} \). Holding \( V_S \) and \( R_S \) constant, let us find the value of \( R_L \) for which \( P_L \) is a maximum.

\[
\frac{dP_L}{dR_L} = 0 = V_S^2 \left[ \frac{1}{(R_L + R_S)^2} - \frac{2R_L}{(R_L + R_S)^3} \right]
\]

\[
\frac{2R_L}{R_L + R_S} = 1
\]

\[
R_L = R_S
\]

Thus matching resistances results in the greatest power transfer to the load.

**Input Resistance**

We have just discussed the fact that a battery, a signal generator or any other source of electrical power has an output resistance. Similarly, the circuit connected to the source, which is called the load, has an input resistance. This is the resistance the generator "sees". In the simple example of the battery and resistor just considered, the input resistance of the load is \( R_L \). Other examples are not quite so trivial.

When two signal devices are connected together, one must pay attention to output and input resistance:
We have already shown that if we want maximum power transfer we should have $R_{01} = R_{i2}$. However, if we want minimum change in the voltage amplitude transferred, it is a different matter. Note that $R_{01}$ and $R_{i2}$ form a voltage divider. If we want maximum signal voltage at the input of the second device, then we must have $R_{i2} \gg R_{01}$.

**Input and Output Resistance of a Voltage Divider: Case A**

Consider the following common circuit:

In this case, the input resistance seen by the generator is given by $R_1$ in series with the parallel resistance of $R_2$ and $R_L$. (If $R_L \gg R_2$, then the input resistance is $R_1 + R_2$.)

The output resistance as seen by $R_L$ is still more complicated. In fact, it is not even obvious that combinations of $R_S$, $R_1$, and $R_2$ with the generator could be described by a simple output resistance. For simplicity, let us include $R_S$ in $R_1$, (with no loss of generality) and call the generator $V$. 
If we replace $R_2$ and $R_L$ by their equivalent parallel resistance, $R_{eq} = \frac{R_2R_L}{R_2 + R_L}$, we have a voltage divider and:

$$V_{out} = \frac{VR_{eq}}{R_{eq} + R_1} = \frac{\frac{VR_2R_L}{R_2 + R_L}}{\frac{R_1 + \frac{R_2R_L}{R_2 + R_L}}{R_2 + R_L}} = \frac{VR_2R_L}{R_1R_2 + R_1R_L + R_2R_L}$$

$$= \frac{VR_2}{R_1 + R_2} \cdot \frac{R_L}{R_L + \frac{R_1R_2}{R_1 + R_2}}$$

which is exactly equivalent to the following circuit for $V' = VR_2/(R_1 + R_2)$ and $R' = R_1R_2/(R_1 + R_2)$:

![Circuit Diagram]

Thus $R_L$ "sees" a circuit with series resistance or output resistance of $R'$, the parallel combination of $R_1$ and $R_2$. The effective battery voltage, $V'$, is less than $V$ since there is a voltage drop across $R_1$ due to the current that flows through $R_1$ and $R_2$ even if $R_L$ were infinite.

**Differential Circuit Parameters**

So far we have considered the so-called D.C. circuit parameters: total current, total voltage, and D.C. resistance (total voltage divided by total current), all indicated by capital letters ($I, V, R$). Now we wish to consider small changes or differential values, and we will use lower case letters to represent them. Thus $v$ is $\Delta V$, $i$ is $\Delta I$, and $r$ is $\Delta I = \Delta V/\Delta I$. For example, we often refer to input and output resistance which are differential values. We will now consider a specific example of great importance.

**Input Resistance of a Voltage Divider: Case B**

In case B, $V$, $R_1$, and $R_2$ are used to establish $V_a$. $V_a$ is, of course, equal to $VR_2/(R_1 + R_2)$. 
If now a small "signal" $i_{in}$ is applied, the potential at the junction of $R_1$ and $R_2$ changes from $V_a$ to $V_a + v_a$, and $I_1$ changes to $I_1 + I_1$. A little thought shows that $I_1$ is negative if $i_{in}$ is positive. We define $r_{in}$ by Ohm's Law, $r_{in} = v_a/i_{in}$. Before proceeding with the calculation of $r_{in}$ we mention that this circuit will appear in an applications context in Lecture 13. We also note in passing that $i_{in}$ is used as the independent variable, i.e., signal, to avoid having a signal generator of voltage $v_a$ load $V_a$. If this statement is confusing don't worry about it. We will also discuss this problem in Lecture 13. We now proceed to find $v_a$ in order to find $r_{in}$. The potential across $R_1$ is:

1. \[ V - (V_a + v_a) = (I_1 + i_1)R_1 \]

The potential across $R_2$ is:

2. \[ V_a + v_a = (I_1 + i_1 + i_{in})R_2 \]

We note that $I_1 + i_1$ can be expressed by a single symbol, $I'$.

Exercise 1

Eliminate $I'$ from equations (1) and (2) and solve for $v_a$. Then substitute for $V_a$ the previously derived value $VR_2/(R_1 + R_2)$.

Ans. \[ v_a = i_{in} \frac{R_1R_2}{R_1 + R_2} \]

Exercise 2

Recall that $r_{in} = v_a/i_{in}$. From the results of the previous exercise show that $r_{in}$ is the equivalent resistance of $R_1$ and $R_2$ in parallel. (For this reason the potential $V$ which was assumed to remain constant is often called a differential ground or an A.C. ground.)

In Exercise 2 we have seen that if a current $i_{in}$ is forced into the junction of $R_1$ and $R_2$, then the potential of that point rises by $v_a$.
For $R_L = \infty$ we instantly have the unloaded voltage divider result of $V_{eq} = VR_2/(R_1+R_2) = V_{eq}$ as before. For $R_L = 0$ we have $I_{short} = V/R_1$. Then:

$$R_{eq} = \frac{V_{open}}{I_{short}} = \frac{VR_2}{V/R_1} = \frac{R_1R_2}{R_1+R_2}$$

as we obtained before in a less straightforward manner.

Sometimes finding a Thevenin equivalent circuit can involve using Thevenin's theorem on a subportion of the circuit. Often a battery and voltage divider will be replaced by a Thevenin equivalent. For example, in exercise 3 below, the battery, the 3.3 kΩ resistor and the 1.8 kΩ resistor form a two-terminal network which is connected to everything to the right at the top and bottom of the 1.8 kΩ resistor, the two terminals. This battery and voltage divider can be replaced by a Thevenin equivalent. The $R_{eq}$ of this equivalent is in series with the 910 Ω resistor, so the $R_{eq}$ of the whole circuit is just the first $R_{eq}+910$ Ω. The $V_{eq}$ of the whole circuit is just the $V_{eq}$ of the first battery and voltage divider. Thus, by knowing the battery and voltage divider Thevenin equivalent, exercise 3 can be solved by inspection.

**Exercise 3**

Determine the Thevenin equivalent circuit (emf and $R_S$) for the circuit inside the dashed rectangle. (Ans. $V_T = 0.529$ volts, $R_T = 2070$ Ω)
Exercise 4

The figure below represents a network with a 10 kΩ (10,000 Ω) potentiometer set so that the two resistances are \( r \times 10 \text{ kΩ} \) and \((1-r) \times 10 \text{ kΩ}\) as shown. Make a sketch of the Thevenin equivalent emf as a function of \( r \).

![Circuit Diagram](image)

 Ans. \( V_T = \frac{1.5 \times 680r}{r(1-r)10,000 + 2880} \)

The plot falls below the proportional-to-\( r \) line due to the \( r(1-r) \) term.

The usual way to draw the circuit would be:

![Simplified Circuit Diagram](image)

The technique of using \( V_{\text{open}} \) and \( I_{\text{short}} \) is not the only way to determine Thevenin equivalent circuits. A technique which is useful in the lab on real circuits involves attaching a variable resistor between the two terminals of the box. The potential difference between the two terminals is measured as the resistor is adjusted. When the voltage is equal to one half the open circuit voltage, \( V_{\text{eq}} \), the resistor is removed and measured with an ohmmeter. Since \( R_{\text{eq}} \) and the adjustable resistor form a voltage divider which divides by two, the adjustable resistor and \( R_{\text{eq}} \) must be of equal size. By measuring the adjustable resistor we have indirectly measured \( R_{\text{eq}} \).
where $v_a = i_{in}R_1R_2/(R_1+R_2)$. If current is drawn out of the junction the potential of the junction drops by the same relationship, $v_a = i_{out}R_1R_2/(R_1+R_2)$. If current is drawn out of the junction then the junction is acting as a source and we see that $R_1R_2/(R_1+R_2)$ is its output resistance. This result is the same one found in Case A.

**Thevenin's Theorem**

In studying Case A above we found that the source, source resistance, $R_1$ and $R_2$ could be replaced by single equivalent source and equivalent source resistance. This example is a special case of Thevenin's Theorem which states that any two terminal box, containing any combination of resistors, pure EMFs, and pure current generators, is equivalent to a two-terminal box containing a single pure EMF in series with a single resistor. Thevenin's Theorem is more than a surprising novelty. The theorem offers a tool for circuit analysis as well as providing an insight into why the concepts of input and output resistances or impedances are useful even for complicated circuits.

If the theorem is to be useful in analysis there must be an easy way to find the equivalent EMF and resistance, and there is. In fact there are several ways, each of which is useful in its own way.

Finding $V_{eq}$ is easiest, and is always done the same way. If $R_L = \infty$, no current is drawn through $R_{eq}$ and the terminal voltage is just $V_{eq}$. Thus the potential difference between the box terminals when there is no load is $V_{eq}$. This measurement may be either theoretical in the analysis of a circuit or a real lab measurement on a real circuit.

If $R_L = 0$ the current drawn is just $V_{eq}/R_{eq} = I_{short}$. $R_{eq}$ is then:

$$R_{eq} = \frac{V_{open}}{I_{short}} = \frac{V_{eq}}{I_{short}}$$

As an exercise let us redo the voltage divider output resistance of Case A using this new tool.
Principle of Superposition

The method of superposition is another powerful technique. It is definitely a theoretical technique for circuit analysis rather than measurement. We will make use of this technique repeatedly in this course. Though useful in finding Thevenin equivalents, the method is of much wider usefulness. We begin by analyzing the following very simple circuit.

I is a current generator; a device which (if ideal) supplies a fixed current regardless of the potential difference across it. What is the Thevenin $V_{eq}$ for this box? The signs on the output define what we mean by a positive $V_{eq}$. Since $I$ must flow through $R$ from top to bottom, the top of $R$ is higher in potential than the bottom by $IR$. The potential difference between the box terminals then is $V_{eq} = -V + IR$. Note that the result is a sum of terms; one for each of the sources, $V$ and $I$. We state without proof that this is a general result for linear circuits; whatever potential difference or current we are solving for is a sum of terms. There is one term for each "source" term. We can find the answer by solving the problem for each source with all the other sources set equal to zero, and then add up the results. (In chapter 14 we will find there are some sources which should never be set = 0. For now don't worry about it.)

The only question left is how do you "set a source to zero" in a circuit? We can answer this question by considering the nature of the two types of sources. What is an ideal emf? It is a device which has a well defined emf between its two terminal regardless of the amount of current flowing through it. What then is an ideal zero emf? It is a short piece of heavy wire; a short circuit. A short circuit has zero potential difference across it regardless of the current through it. So to replace a battery by one of zero emf we replace it with a short. We say we have "suppressed" the battery.

How do we suppress a current generator? Again consider the nature of the source. A current source has a fixed current through it regardless of the potential difference between its terminals. So a zero current generator is an open circuit since an open circuit will have zero current through it regardless of the potential difference between its terminals. We suppress a current generator by replacing it with an open circuit.
Let us find $V_{eq}$ for our simple example with superposition.

$$V_{eq} = V_{eq1} + V_{eq2} = -V + IR$$

As another use of superposition let us apply it to a Thevenin equivalent circuit of $V_{eq}$ and $R_{eq}$. If we suppress $V_{eq}$ what do we have left? We just have $R_{eq}$ connected to the two terminals. So one way to find the $R_{eq}$ of some complex circuit is to just suppress all sources and find the $R_{eq}$ of the resulting circuit. The following problem is particularly nice to do by superposition, and is an important case.

**Exercise 5**

Find the Thevenin equivalent emf and resistance for the following circuit:

$$V_T = \frac{V_1 R_2 + V_2 R_1}{R_1 + R_2} \quad R_T = \frac{R_1 R_2}{R_1 + R_2}$$

Often in circuits we have two resistors connected together with a different potential at each end, and we want to know the voltage at the point where they are connected together. The results of exercise 5, which by superposition and inspection, is the sum of the two voltage divider results, one for each source, is applicable in these circuits.