LECTURE 11: COMPLEX IMPEDANCES

Resistance is one kind of impedance. Of course, with a resistor the relationship of voltage to current is given by:

\[ V = IR \]

This is such a useful relationship that we wish to generalize it to any sort of passive network. The definition of the impedance of a given network is that it is whatever is required to make the following relationship correct:

\[ V = IZ \]  \hspace{1cm} (1)\]

where \( Z \) is the impedance of the network. Of course, if the network consists of a single resistor, \( Z = R \). We will succeed in finding expressions for \( Z \) such that equation \( (1) \) will be correct for networks also containing capacitors and inductors, but it will take some doing. To illustrate the difficulties, let us consider a network consisting of a single inductor and then one consisting of a single capacitor.

The voltage across an inductor is given by:

\[ V = L \frac{dI}{dt} \]

We also want it to be true that:

\[ V = IZ_L \] \hspace{1cm} (2)\]

Setting the two equal:

\[ L \frac{dI}{dt} = IZ_L \]

\[ \frac{dI}{I} = \frac{Z_L}{L} \, dt \]

\[ \ln I = \left( \frac{Z_L}{L} \right) t + \ln I_0 \]

\[ I = I_0 e^{Z_L t} \]
Now if $Z_L$ is a real positive number, the current must increase exponentially. If $Z_L$ is a real negative number, it must decrease exponentially toward zero. If $Z_L$ is zero, the voltage is identically zero from equation (2). Since none of these possibilities is general enough to be of much use, we must seek other possibilities. One possibility is to let $Z_L$ vary with time. But a time varying impedance is not a useful concept for a simple inductor. The best solution is to move out of the domain of real numbers and explore the possibility of letting $Z_L$ be imaginary or possibly complex. We will do so shortly, but first let us examine the capacitor.

![Circuit Diagram](image)

The voltage across a capacitor is given by:

$$V = \frac{Q}{C}$$

We also want it to be true that:

$$V = \frac{1}{2} Z_C I$$

Setting the two equal:

$$\frac{Q}{C} = \frac{1}{2} Z_C I = Z_C \left( \frac{dQ}{dt} \right)$$

$$\frac{dQ}{Q} = \frac{dt}{Z_C C}$$

$$\ln Q = (t/Z_C C) + \ln Q_0$$

$$Q = Q_0 e^{t/(Z_C C)}$$

$$I = \frac{Q_0}{Z_C C} e^{t/(Z_C C)} = I_0 e^{t/(Z_C C)}$$

This relationship is so similar to the one we obtained for the inductor, that we can immediately draw a similar conclusion: We must allow $Z_C$ to be imaginary or possibly complex.
Complex Exponentials

With both the capacitor and the inductor, we obtained currents of the form \( I = Ke^{At} \) where \( A \) is imaginary, or possibly complex. If \( A \) is complex then \( A = B + jD \), where \( j = \sqrt{-1} \). \( I \) can be written in the form \( I = Ke^{Bt}e^{jDt} \). \( e^{Bt} \) is either constantly increasing or constantly decreasing. We will expect, then, that usually \( B = 0 \). Thus we are led to the form \( I = K e^{jDt} \). Let us examine this expression more fully. In case you have seen all this before we have boxed the important results.

The usual expansions for \( e^x, \sin x, \) and \( \cos x \) are:

\[
e^x = 1 + x/1! + x^2/2! + x^3/3! + ... \\
\sin x = x - x^3/3! + x^5/5! - x^7/7! + ... \\
\cos x = 1 - x^2/2! + x^4/4! - x^6/6! + ...
\]

Now if we assume that the expansions hold for imaginary numbers (or define things so that they must hold), we find:

\[
e^{jx} = 1 + jx/1! + j^2x^2/2! + j^3x^3/3! + j^4x^4/4! + ...
\]

Now, \( j = \sqrt{-1}, \ j^2 = -1, \ j^3 = -j, \ j^4 = +1, \ldots, \) so:

\[
e^{jx} = 1 + jx/1! - x^2/2! - jx^3/3! + x^4/4! + jx^5/5! + ... \\
e^{jx} = (1 - x^2/2! + x^4/4! ... ) + j(x - x^3/3! + x^5/5! + ... )
\]

\[
e^{jx} = \cos x + j \sin x
\]

Now let us check to see that the usual expression:

\[
e^{jx_1}e^{jx_2} = e^{j(x_1 + x_2)}
\]

is true:

\[
e^{jx_1}e^{jx_2} = (\cos x_1 + j \sin x_1)(\cos x_2 + j \sin x_2)
\]

\[
e^{jx_1}e^{jx_2} = (\cos x_1 \cos x_2 - \sin x_1 \sin x_2) + j(\cos x_1 \sin x_2 + \sin x_1 \cos x_2)
\]

\[
e^{jx_1}e^{jx_2} = \cos(x_1 + x_2) + j \sin(x_1 + x_2)
\]

\[
e^{jx_1}e^{jx_2} = e^{j(x_1 + x_2)}
\]
Complex Currents

A single, pure frequency is often written \(\cos(\omega t + \delta)\). (Recall that \(\omega = 2\pi f\) where \(f\) is the frequency.) For the time being, let us consider the case where \(\delta = 0\). \(\cos \omega t\) is just the real part of \(e^{j\omega t}\). Thus we can say that any pure sine-wave-type current is just the real part of a current of the form \(I = I_0 e^{j\omega t}\). As a convention we will drop the expression "real part of" and regard the current as \(I_0 e^{j\omega t}\). However, remember that real instruments measure only real currents or voltages!

Any repeated waveform can be regarded as being composed of a sum of sine waves, the fundamental plus its harmonics. Similarly, a single pulse is composed of a continuous set of sine waves. Thus we can apply the present analysis to various waveforms.

Complex Impedances

Now let us derive the impedance of a capacitor and of an inductor at a single frequency. First the inductor:

\[
L \frac{dI}{dt} = jIZ_L \quad \text{with} \quad I = I_0 e^{j\omega t}
\]

\[
L \frac{d}{dt} (I_0 e^{j\omega t}) = I_0 e^{j\omega t} Z_L
\]

\[
L I_0 e^{j\omega t} = I_0 e^{j\omega t} Z_L
\]

\[
\omega L = Z_L
\]

You will note that the impedance depends on the frequency. Thus different harmonics of a periodic waveform will encounter different impedances.

Similarly, for a capacitor:

\[
\frac{Q}{C} = jIZ_C
\]

\[
\frac{1}{C} \int I \, dt = jIZ_C
\]

\[
\frac{1}{C} \int I_0 e^{j\omega t} \, dt = I_0 e^{j\omega t} Z_C
\]

\[
\frac{I_0 e^{j\omega t}}{Cj\omega} = I_0 e^{j\omega t} Z_C
\]
\[
\frac{1}{j\omega C} = Z_C
\]

We now note that as frequency increases, the impedance of an inductor increases while that of a capacitor decreases.

**Argand Diagram**

We usually plot complex numbers on a cartesian system with the real part on the horizontal axis and the imaginary part on the vertical axis.

The exponential expressions for complex numbers are then equivalent to a polar coordinate system.

\[
Re^\theta = R\cos\theta + jR\sin\theta = X + jY
\]

\[Z = j\omega L\] plots as:

so another way to express this \( Z \) is as \( \omega L e^{j\pi/2} \). \( I = I_0 e^{j\omega t} \) plots as:
The projection of $I_0$ on the real axis, which is the real part of $I$, is just $I_0 \cos \omega t$.

The voltage across the inductor which carries current $I$ is $V=IZ_L$, which is:

$$I_0 e^{j\omega t} \omega L e^{j\pi/2} = I_0 \omega L e^{j(\omega t + \pi/2)}$$

The projection of $V$ on the real axis is:

$$I_0 \omega L \cos(\omega t + \pi/2) = I_0 \omega L [\cos \omega t \cos \pi/2 - \sin \omega t \sin \pi/2] = -I_0 \omega L \sin \omega t$$

The current through the inductor lags the voltage across the inductor by 90°.

**THE FACT THAT THE VOLTAGE AND CURRENT MAY NOT BE IN PHASE FOR CIRCUITS CONTAINING INDUCTORS AND CAPACITORS IS WHAT FORCES US TO USE COMPLEX IMPEDANCES.**

**Real or Complex?**

There is often a question as to when to take the real part of a complex expression and when to take the whole complex expression. In general the relationships we are involved with have the form:

$$Output = (\text{Transfer Characteristic})(Input)$$

For example:

$$V = (Z)(I)$$

or:

$$V_{out} = (\text{Gain})(V_{in})$$
In principle, what we must do is express the "Input" in complex form (e.g. \( I = I_0e^{j\omega t} \) above), express the "Transfer Characteristic" in complex form (e.g. \( Z = j\omega L \) above), find the complex form of the "Output" (\( I_0\omega L e^{j(\omega t + \pi/2)} \) above), and finally find the real part of the "Output" (-\( I_0\omega L \sin\omega t \) above). Usually we do not do all these steps. In particular, we often only find the complex transfer characteristic knowing that the above procedure is implied.

**Example: Series Impedances**

Now consider a resistor in series with an inductor. By the same arguments as used in Chapter 1, the same I must flow through both elements.

\[
V_b = IZ_L = Ij\omega L
\]

\[
V_a = V_b + IR = Ij\omega L + IR = I(R + j\omega L)
\]

\[
V_a = IZ_{L,R} = I_0e^{j\omega t}\sqrt{R^2 + \omega^2 L^2} \cos \phi, \text{ where } \tan \phi = \omega L/R
\]

\[
Z_{L,R} = \sqrt{R^2 + \omega^2 L^2} \cos \phi
\]

Consider a resistor, capacitor and inductor:

\[
V = I \left[ R + j\omega L + \frac{1}{j\omega C} \right] = I \left[ R + j\left(\omega L - \frac{1}{\omega C}\right) \right]
\]

\[
Z_{R,L,C} = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \cos \phi
\]
\[ \tan \phi = \frac{\omega L - \frac{1}{\omega C}}{R} \]

With complex impedances, all the rules for series impedance, parallel impedance, and voltage dividers will be the same as they were with resistors, since the derivations depended only on the equation \( V = IR \), and we now have \( V = IZ \).

**Example: Voltage Divider**

\[ V = IZ_1 + IZ_2 = I(Z_1 + Z_2) \]

so \( I = \frac{V}{Z_1 + Z_2} \)

\[ V_a = IZ_2 = \frac{VZ_2}{Z_1 + Z_2} \]

Now suppose \( Z_1 \) is a resistor \( R \) and \( Z_2 \) is a capacitor \( C \).

\[ V_a = V \left[ \frac{Z_2}{Z_1 + Z_2} \right] = V \left[ \frac{1}{\frac{1}{j\omega C} + \frac{1}{R}} \right] \]

\[ V_a = \frac{V}{1 + j\omega RC} \]

Use:

\[ 1 + j\omega RC = \sqrt{1 + \omega^2 R^2 C^2} \quad e^{j\phi} \text{ with } \phi = \tan^{-1}(\omega RC) \]

Then:

\[ V_a = \frac{V}{\sqrt{1 + \omega^2 R^2 C^2} e^{j\phi}} \]

\[ = \frac{V e^{-j\phi}}{\sqrt{1 + \omega^2 R^2 C^2}} \]

In addition to a phase shift (\( V_a \) lags \( V \) by \( \phi \)), the amplitude is reduced by a factor \( \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}} \).
Example: Operational Amplifier with Complex Feedback

By virtual equality: \( V_S = V_o \frac{Z_1}{Z_1 + Z_2} \)

\[
\text{gain} = \frac{V_o}{V_S} = \frac{Z_1 + Z_2}{Z_1}
\]

If \( Z_2 \gg Z_1 \) the gain is \( \approx Z_2/Z_1 \). The inverting configuration can be treated similarly. In general, the gain involves a magnitude ratio and a phase shift.

Exercise 1

Design an amplifier which will have a gain magnitude inversely proportional to the frequency within 10% over the range 100 Hz to 1 kHz.

(Ans: By virtual equality an integrator [Chapter 4] has gain \(-1/\omega CR\). C and R are chosen by gain and input impedance considerations. For virtual equality the circuit gain must be \(< 10^3\) at 1 kHz assuming we use an LF351.)

Exercise 2

Plot the absolute value of impedance versus frequency for the following circuit:

\[
1 \text{ mH} \quad C = 0.1 \ \mu \text{f}
\]

Hint: It would be good to show frequency on a logarithmic plot. Be sure that the plot includes the frequency 15.9 kHz.

(Ans: For frequencies much less than 15.9 kHz the slope is plus one. For frequencies much greater than 15.9 kHz the slope is minus one. At 15.9 kHz the impedance rises to infinity for pure L and pure C. This behavior at 15.9 kHz is called parallel resonance.)