LECTURE 14: ACTIVE FILTERS

There are many occasions when good research demands a frequency-selective filter. They can be made in three general types:

Filter

- **Low Pass**: Doesn't affect sine waves with \( \omega < \omega_L \).
  - Severely attenuates sine waves with \( \omega > \omega_L \).

- **High Pass**: Doesn't affect sine waves with \( \omega > \omega_H \).
  - Severely attenuates sine waves with \( \omega < \omega_H \).

- **Band Pass**: Doesn't affect sine waves with \( \omega_L < \omega < \omega_H \).
  - Severely attenuates sine waves with \( \omega > \omega_H \) or \( \omega < \omega_L \).

Of course, no system is really discontinuous at \( \omega_L \) or \( \omega_H \). One simply approximates the characteristics of a given filter. These filters find application in a wide variety of studies: in psychology to separate \( \alpha \) brain waves from \( \theta \) waves; in geology for detection of particular modes of seismic disturbances; in radios to tune in a particular station; in mechanical engineering to select particular vibrations; in anti-submarine warfare to select sonar reflections.

Filtering can be done passively with capacitors, resistors and inductors. Such an approach often involves expensive, bulky components; furthermore the signal is often attenuated significantly. The use of op amps permits inexpensive, small, effective filters, particularly at low frequencies.

For a description of a large variety of active filters, along with design procedures, see Operational Amplifiers by Burr-Brown Engineering staff, (McGraw-Hill, Inc., 1971). In this book, as in many engineering texts, "s" is used for \( \omega \). A simpler discussion is available in The Active-Filter Cookbook by Dan Lancaster (Howard Sams, 1975).

We will illustrate the possibilities with one practical example, the active filter with a multiple feedback network.
By setting $V_v = 0$, we get the following equations:

\[
V_B = V_S - I_1 Z_1
\]

\[
V_B = (I_1 + I_2 + I_3) Z_2
\]

\[
V_o = I_3 Z_5
\]

\[
V_o - V_B = I_2 Z_4
\]

\[
V_B = -I_3 Z_3
\]

**Exercise 1**

Eliminate $V_B$, $I_1$, $I_2$ and $I_3$ from the above equations and solve for $V_o/V_S$.

\[
\begin{bmatrix}
V_o \\
V_S
\end{bmatrix}
= \frac{Z_2}{Z_1}
\left[
\frac{Z_2}{Z_5} + \frac{Z_3 Z_2}{Z_1 Z_5} + \frac{Z_2}{Z_5} + \frac{Z_2}{Z_4} + \frac{Z_2 Z_3}{Z_4 Z_5}
\right]
\]

which can be rewritten as:

\[
\frac{V_o}{V_S} = -\frac{Z_2 Z_4 Z_5}{Z_1 Z_2 Z_3 + Z_1 Z_2 Z_4 + Z_1 Z_2 Z_5 + Z_1 Z_3 Z_4 + Z_2 Z_3 Z_4}
\]

(1)

**Low Pass Filter**

\[
\begin{align*}
Z_1 &= R_1 \\
Z_2 &= \frac{1}{j\omega C_2} \\
Z_3 &= R_3 \\
Z_4 &= R_4 \\
Z_5 &= \frac{1}{j\omega C_5}
\end{align*}
\]

(2)

Before we calculate, let us examine the performance qualitatively: At very low frequencies, $C_2$ and $C_5$ represent infinite impedance. For such cases, the equivalent circuit is:
For such a system the gain is $-R_4/R_1$. In the ideal case $R_3$ plays no role since the op amp input neither sources nor sinks current. If $R_3$ is not high it can be neglected in actual use.

At very high frequencies, $C_2$ and $C_5$ represent short circuits. Thus $V_p$ is zero. Furthermore, the output is connected to $V_o$ which is at virtual ground. Thus the gain is expected to approach 0 as $1/\omega^2$ as frequency becomes very large.

Now let us plug the impedances of Eq. (2) into Eq. (1) resulting in Eq. (3).

\[
\frac{V_o}{V_s} = -\frac{R_4}{j\omega C_2 R_3 C_5} + \frac{R_4}{j^2 \omega^2 C_2 C_5} + R_4 + R_4 + \frac{R_4}{j\omega C_2}
\]

Multiplying numerator and denominator by: $j^2 \omega^2$

\[
\frac{V_o}{V_s} = -\frac{1}{R_4 C_2 R_3 C_5 + \frac{j\omega}{R_4 C_2} + \frac{j\omega}{R_3 C_5} + \frac{j\omega}{R_1 C_2} + \frac{j\omega}{R_4 C_5}}
\]

\[
\frac{V_o}{V_s} = -\frac{1}{R_4 R_3 C_2 C_5 - \omega^2 + j\omega C_2 \left(\frac{1}{R_1} + \frac{1}{R_3} + \frac{1}{R_4}\right) + \frac{1}{R_3 R_4 C_2 C_5}}
\]

(3)

**Exercise 2**

Show that the above equation can be written in the form:

\[
\frac{V_o}{V_s} = -\frac{H_o \omega_o^2}{-\omega^2 + j\omega \alpha \omega_o + \omega_o^2}
\]

(This is the standard form for the "complex-conjugate-pole-pair low-pass transfer function.") Here:

\[
H_o = \frac{R_4}{R_1}, \quad \omega_o = \left(\frac{1}{R_3 R_4 C_2 C_5}\right)^{1/2} \quad \text{and} \quad \alpha = \sqrt{\frac{C_5}{C_2} \left(\frac{1}{R_3 R_4 C_2 C_5} + \frac{R_3}{R_4} + \frac{\sqrt{R_3 R_4}}{R_1}\right)}
\]
For $\omega \gg \omega_0$, $|V_o/V_s| \rightarrow H_0(\omega_0/\omega)^2$ and that for $\omega \ll \omega_0$, $|V_o/V_s| \rightarrow H_0 = R_4/R_1$ as we predicted. Note also that at low frequency $V_o$ and $V_s$ are $180^\circ$ out of phase. At $\omega = \omega_0$ (where the $j$ term is important) $V_o$ leads $V_s$ by $90^\circ$.

Magnitude response of second-order low-pass filters for selected $\alpha$ values $\leq 2$.

High Pass Filter

Once again, let us first determine the response qualitatively. At low frequencies, the capacitive impedances greatly exceed the resistive ones, so at point B, the input is attenuated to near zero. Furthermore, the output is at the same potential as $V_s$, that is, near ground. Thus the gain goes to zero at low frequencies as $\omega^2$. At high
frequencies, the capacitive impedances are much lower than the resistive ones and the equivalent circuit is:

\[ V_S \quad \begin{array}{c}
\downarrow \\
C_1 \\
\uparrow \\
\downarrow \\
C_3 \\
\uparrow \\
C_4
\end{array} \quad \frac{1}{j\omega C_4} \quad \frac{1}{j\omega C_1} \quad \frac{1}{j\omega C_2 C_4} \quad \frac{1}{j\omega C_3 C_4}

The gain in this configuration is \(-\frac{Z_4}{Z_1}\) or:

\[
\frac{V_o}{V_S} = -\frac{1}{j\omega C_4} = -\frac{C_1}{C_4}
\]

Now let us calculate the response more carefully. Plugging Eqs. (5) in Eq. (1):

\[
\frac{V_o}{V_S} = -\frac{R_2 R_5}{R_2 \frac{1}{j\omega C_4}} + \frac{R_2 R_5}{R_3 C_4} + \frac{R_5}{j\omega C_1} + \frac{1}{j\omega C_2 C_4} + \frac{1}{j\omega C_3 C_4}
\]

Multiplying numerator and denominator by: \(R_2 R_5\)

\[
\frac{V_o}{V_S} = -\frac{j\omega^2 C_1}{C_4}
\]

Rearranging:

\[
\frac{V_o}{V_S} = -\frac{-\omega^2 C_1}{C_4} + \frac{1}{R_2 C_3} + \frac{1}{R_5 C_4} + \frac{1}{R_5 C_3 C_4} + \frac{1}{R_2 R_5 C_3 C_4}
\]

**Exercise 3**

Show that the above equation can be written in the form:

\[
\frac{V_o}{V_S} = -\frac{-H_0 \omega^2}{\omega^2 + \alpha \omega \omega_0 + \omega_0^2}
\]

(This is the standard form for the "complex-conjugate-pole-pair high-pass transfer function.") Here:

\[
H_0 = \frac{C_1}{C_4}, \quad \omega_0 = \left(\frac{1}{R_2 R_5 C_3 C_4}\right)^{\frac{1}{2}} \quad \text{and} \quad \alpha = \frac{R_2}{R_5} \left[\frac{C_1}{\sqrt{C_3 C_4} + \sqrt{C_3 + C_4}}\right]
\]
For \( \omega \gg \omega_0 \), \( |V_o/V_s|_H \to H_0 = C_1/C_4 \) and that for \( \omega \ll \omega_0 \), \( |V_o/V_s|_L \to H_0(\omega/\omega_0)^2 \). These results confirm our expectations.

Note also that \( |V_o/V_s|_H \) is the same as \( |V_o/V_s|_L \) with \( \omega \) and \( \omega_0 \) interchanged. Thus the graph of \( |V_o/V_s|_L \) as a function with \( \omega/\omega_0 \) is the same as \( |V_o/V_s|_H \) if we change \( \omega/\omega_0 \) to \( \omega_0/\omega \).

One can get a good bandpass filter by placing an appropriately-chosen high-pass filter in series with an appropriately-chosen low-pass filter. The reference given at the beginning of this section also gives single op amp bandpass circuits.

For both the filters we worked out, it turns out that good design requires \( H_0/\alpha \leq 100 \).

**Design Procedure**

We must choose the values for the five Rs and Cs involved in our filters. Since there are only three parameters (\( H_0 \), \( \omega_0 \) and \( \alpha \)) which characterize the filter we have an underdetermined problem. Even when we add the condition that the input impedance must be greater than some value, we have too much freedom. One approach is to start with values for two of the Cs which are convenient in terms of our parts kit and capacitor physical sizes. We see what the other components must be for our choice of \( H_0 \), \( \omega_0 \) and \( \alpha \). If these other values are not reasonable (too big or too small) then we start the procedure over again.

There are other procedures. For example for the low pass filter we can simplify the situation by requiring \( R_1 = R_3 = R_4 = R \). The cost of this choice is that \( H_0 \) is forced to be 1. The simplified expression for \( \alpha \) and \( \omega_0 \) are:

\[
\alpha = \frac{3 \sqrt{C_3}}{\sqrt{C_2}}
\]

\[
\omega_0 = \frac{1}{R} \sqrt{\frac{1}{C_2 C_5}}
\]

Our second simplification is to introduce C and relate \( C_2 \) and \( C_5 \) to it by:

\[
C_2 = \frac{3C}{\alpha}
\]

\[
C_5 = \frac{\alpha C}{3}
\]

This choice is consistent with the \( \alpha \) expression above and that:

\[
\omega_0 = \frac{1}{RC}
\]
We pick $R$ by the input impedance desired, and the other choices follow from the required $\omega_0$ and $\alpha$. 