Formulae

\[ I = \varepsilon_0 c \left( \frac{E^2}{\lambda} \right) = \frac{1}{2} \varepsilon_0 c E_0^2 \]

\[ \varepsilon_r = \hbar \nu \]

\[ n = c/\nu \]

\[ \nu = \lambda \nu \]

\( (\nu = \text{"vee"}; \nu = \text{"nu"}.) \)

\[ k = 2\pi/\lambda \]

\[ \omega = 2\pi \nu \]

\[ \tau = 1/\nu \]

\[ D = \frac{1}{f} = (n - 1) \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \]

\[ D_{\text{tot}} = D_1 + D_2 \quad \text{(thin lenses in contact)} \]

\[ \frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f} \]

\[ M_T = \frac{y_i}{y_o} = -\frac{s_i}{s_o} \]

\[ n_1 \sin \theta_1 = n_2 \sin \theta_2 \]

\[ \theta_r = \theta_i \]
Problem 1 (25 points)
A very simplified model for an eye is a solid sphere of radius $R$ and index of refraction $n_e$, with an aperture on one side and a retina on the opposite side, as shown in the diagram below. Assume that the index of refraction $n = 1$ for the medium outside the eye, and also assume that all angles in this problem are small enough to use the "small angle" trigonometric approximation.

(a) The diagram shows a "chief ray" from an object, which enters the center of the eye aperture and makes a small angle $\alpha$ with the optical axis outside the eye. In terms of $\alpha$, $R$, and $n_e$, at what height $y_i$ above the optical axis does this ray strike the retina? Please observe the convention that $y_i > 0$ above the optical axis, and $y_i < 0$ below the optical axis. (Hint: What angle does the ray make with the optical axis inside the eye?)

Snell's law at interface: \[ \sin \alpha = n_e \sin \beta \]
\[ \Rightarrow \alpha = n_e \beta \Rightarrow \beta = \frac{\alpha}{n_e} \]
\[ \frac{y_i}{2R} \approx \beta = \frac{\alpha}{n_e} \Rightarrow \boxed{y_i = \frac{2R\alpha}{n_e} \text{ or } y_i = -\frac{2R\alpha}{n_e}} \]

(b) The relationship between image and object distances for a single spherical interface with radius of curvature $R$ between a medium with index of refraction $n_1$ and a medium with index $n_2$ is given by
\[ \frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R} \]
for an object in medium $n_1$ and the center of curvature within medium $n_2$.

Using this relation, find the value for this simple eye's index of refraction $n_e$ that would be needed to focus rays from $s_o = \infty$ onto the retina.

\[ \frac{1}{\infty} + \frac{n_e}{2R} = \frac{n_e - 1}{R} \Rightarrow \frac{1}{2} n_e = n_e - 1 \]
\[ \Rightarrow \frac{1}{2} n_e = 1 \Rightarrow \boxed{n_e = 2} \]
Problem 2 (25 points)
A star has a luminosity $L$ (this has units of power: i.e., energy per unit time) and a radius $R$. A planet orbits the star in a circular orbit at a distance $d$ from the star. The star emits light across a spectrum of wavelengths, but you may assume that all emission is at an effective wavelength of $\lambda_{\text{eff}}$.

Provide answers to the following three questions in terms of $L$, $R$, $d$, $\lambda_{\text{eff}}$, and fundamental constants:

(a) What is the electromagnetic energy density (energy per unit volume) in the space just above the surface of the star? (I.e., at a height that is negligible in comparison to $R$.)

\[
\text{Irradiance is } \frac{L}{4\pi R^2}, \quad (= I)
\]

Energy density (see above) is

\[
\nu = \frac{d\text{(Energy)}}{dV} = \frac{d\text{(Energy)}}{cdt \, dA} = \frac{I}{c} = \frac{L}{4\pi R^2 c}
\]

(b) What is the flux of photons per unit area per unit time from the star at the location of the planet?

Energy flux (Irradiance) is $\frac{L}{4\pi d^2}$.

Energy per photon is $E_\nu = h\nu_n = hc/\lambda_{\text{eff}}$

Photon flux is then

\[
\frac{dN_\lambda}{dt} = \frac{I}{E_n} = \frac{L \lambda_{\text{eff}}}{4\pi h c d^2}
\]

(c) What is the ratio of the time-averaged electric field amplitude at the location of the planet to the time-averaged electric field amplitude just above the surface of the star?

2 possible answers (ambiguous wording):
1) Ratio of irradiance is $d^2/R^2$ by energy cons., and $I \sim E^2$, so field ratio is $d/R$
2) $<E^2> = 0$, so answer undefined (or zero...)
Problem 3 (25 points)
If two materials have indices of refraction that differ by only a small amount, the critical angle for total internal reflection will be close to 90° for rays traveling from the higher-index medium across an interface into the lower-index medium.

Consider the following diagram, consisting of a “sandwich” of dielectric slabs, with the inner slab having index of refraction \( n_s \) and the outer slabs having index of refraction \( (1 - \delta)n_s \), where \( \delta << 1 \). The slabs are surrounded by vacuum with \( n = 1 \). A ray traverses the system as shown in the diagram. Assume that \( \theta_{i1} \) and \( \theta_{t1} \) are small angles, but do **not** assume that \( \theta_{i2} \) and \( \theta_{t2} \) are small angles.

(a) Express Snell’s law for the first two interfaces encountered by the ray: from \( n = 1 \) to \( n = n_s \), then from \( n = n_s \) to \( n = (1 - \delta)n_s \).

1) \( \theta_{i1} = n_s \theta_{t1} \quad \text{or} \quad \sin \theta_{i1} = n_s \sin \theta_{t1} \)

2) \( n_s \sin \theta_{i2} = (1 - \delta)n_s \sin \theta_{t2} \)

(b) Now set \( \theta_{t2} = 90° \), corresponding to total internal reflection at the second interface. Using the fact that \( \sin \theta_{i2} = \cos \theta_{t1} \), and the approximation that \( \cos \theta \simeq 1 - \theta^2/2 \) for small \( \theta \), derive an approximate expression for \( \theta_{t1} \) that does not involve trigonometric functions.

\[
\sin \theta_{i2} = \cos \theta_{t1} \simeq 1 - \frac{1}{2} \theta_{t1}^2 = (1 - \delta)
\]

\[
\Rightarrow \quad \theta_{t1} = \sqrt{2\delta}
\]

(c) Finally, use the approximation that \( \sin \theta \simeq \theta \) for small \( \theta \) to derive an expression for \( \theta_{i1} \) in terms of \( n_s \) and \( \delta \) that does not involve trigonometric functions. This is the maximum input angle for rays to be confined and “guided” within the central slab.

\[
\theta_{i1} = n_s \theta_{t1} = \sqrt{n_s \sqrt{2\delta}}
\]
Question 1: (5 points) Match the Maxwell equations to their names:

(a) Ampere’s Law
(b) Faraday’s Law
(c) Gauss’s Law
(d) No magnetic monopoles

(i) \[ \mathbf{\nabla} \cdot \mathbf{E} = \frac{1}{c_0^2} \iiint \rho \, dV \]
(ii) \[ \mathbf{\nabla} \cdot \mathbf{B} = 0 \]
(iii) \[ \oint \mathbf{E} \cdot d\ell = -\iint (\partial \mathbf{B} / \partial t) \cdot d\mathbf{S} \]
(iv) \[ \oint \mathbf{B} \cdot d\ell = \iint (\mu_0 \mathbf{J} + \mu_0 \varepsilon \partial \mathbf{E} / \partial t) \cdot d\mathbf{S} \]

(object on (a) = (civ))

Question 2: (5 points) Identify each of the 4 aberrations represented in the following figure:

- Distortion
- Astigmatism
- Coma
- Spherical

This will also accept chromatic for this one.

Question 3: (5 points) Write an expression for the time- and space- dependent electric field \( \mathbf{E}(x, y, z, t) \) for a harmonic light wave of frequency \( \nu \), traveling in the positive \( x \)-direction within a medium of index of refraction \( n \), with the electric field oriented in the \( z \)-direction, with an electric field amplitude \( E_0 \).

\[
\mathbf{E}(x, y, z, t) = E_0 \hat{z} \sin \left[ 2\pi \nu \left( \frac{n}{c} x - t \right) \right] \quad \text{(e.g.,)}
\]

(0, \( x \) to have in terms of substitutions, if defined.)

Question 4: (5 points) Rearrange the following list of electromagnetic bands to put them in order of increasing frequency: (1) visible light, (2) radio waves, (3) microwaves, (4) gamma rays, (5) infrared light, (6) ultraviolet light, (7) X-rays.

(1) visible, (2) radio, (3) microwaves, (5) infrared, (6) ultraviolet, (7) X-rays, (4) gamma rays

Question 5: (5 points) A thin positive lens of focal length \( f_1 = +20 \) cm is placed in close contact (negligible separation) with a thin negative lens of focal length \( f_2 = -100 \) cm. An object is then located 20 cm along the optical axis from the lens combination. How far from the lenses does the image of the object form? Is the image real or virtual?

\[
f = \frac{1}{f_1} + \frac{1}{f_2} \quad \frac{1}{20} + \frac{1}{-100} = \frac{5}{100} - \frac{1}{100} = \frac{4}{100} \Rightarrow f_{\text{comb}} = 25 \text{ cm}.
\]

\[
\frac{1}{20} + \frac{1}{s_i} = \frac{1}{25} \Rightarrow s_i = \frac{4}{100} \left( -\frac{5}{100} \right) \Rightarrow s_i = -100 \text{ cm (virtual image)}
\]