1. The number of counts during a 1-second measurement interval fluctuates from measurement to measurement and is distributed according to Poisson statistics:

\[ P(n, m) = \frac{m^n e^{-m}}{n!} \]

where \( m \) is the average number of counts and \( n \) is the number of counts, probability of which we are interested in.

Thus, we have \( m = 3.5 \); \( n = 2 \) and the probability to register 2 counts while the average count is 3.5 is:

\[ P(2, 3.5) = \frac{3.5^2 e^{-3.5}}{2!} \approx 0.185 \]

If we make one long measurement, that gives \( n \) counts, our best estimate of average \( m \) is \( n \), and consequently, the best estimation for the standard deviation \( \sqrt{m} \) is \( \sqrt{n} \).

\[ \sqrt{n} = \sqrt{4096} = 64 \]  

This error is accumulated through 5 minutes, hence per second we obtain \( \frac{4096}{5.60} \approx 13.65 \) counts with error \( \frac{64}{5.60} \approx 0.21 \): \( 13.65 \pm 0.21 \) counts/s. Since the number of counts is very large, we can approximate the Poisson distribution with Gaussian one (again, we use \( n \) as our best approximation for actual average number of counts \( m \)); probability to observe \( N \) counts is:

\[ P(N) = \frac{1}{\sqrt{2\pi m}} e^{-\frac{(N-n)^2}{2n}}, \]

where we used
The area under the
P(N) curve within the interval
\([n-\sigma, n+\sigma]\) (or the sum of probabilities to observe any value of N within this interval) is \(\approx 0.683\).

Thus the probability to measure \(N\) outside of this interval is \(1 - 0.683 \approx 0.32\). Since the Gaussian distribution is symmetric, the probabilities of large fluctuations towards large or small \(N\) are the same. Hence the probability that the measurement is low by more than an error is \(\frac{1}{2} \times 0.32 = 0.16 = 16\%\). Of course this probability equally applies to the "per second" rate derived from measurement of \(n\).

\(2\)

Form factor is given by

\[ F(q) = \frac{1}{2\pi} \int \rho(r) \, d^3r \, e^{iqr/\hbar} \quad (\text{see Williams}) \]

Now we assume that all the charge is concentrated on a surface: \(\rho(r) = \frac{Ze \cdot 8(r-a)}{4\pi a^2}\), where \(a\) is the radius of the nucleus (check that \(\int \rho(r) \, dv = Ze\)). Since the problem is spherically symmetric, we can choose any of the basis set. Let axis \(z\) be directed along the vector \(\vec{q}\), then \(\vec{q} \cdot \vec{r} = qr \cos \theta\) in polar coordinates.
The integral can be now rewritten in polar coordinates as

\[ F(q) = \frac{1}{2\pi} \int \rho(r) r^2 dr \cdot \sin \theta \cdot \cos \theta \cdot e^{\frac{iq \cos \theta}{\hbar}} = \]

no \(\theta\)-dependence, the integral is trivial (2\pi)

\[ = \frac{2\pi i q \rho}{4 \pi a^2} \int \frac{e^{-i q \cos \theta / \hbar}}{r^2} \cdot \sin \theta \cdot \cos \theta \cdot e^{i q \cos \theta / \hbar} \cdot \frac{1}{i q / \hbar} \]

\[ = \frac{2\pi i q \rho}{4 \pi a^2} \int \delta(r-a) \frac{1}{i q / \hbar} \cdot \sin \frac{q x}{\hbar} \]

\[ = \sin \frac{q a}{\hbar} \cdot \frac{1}{i q / \hbar} \cdot \int \delta(r-a) \frac{1}{i q / \hbar} \]

\[ \frac{q a}{\hbar} = \begin{cases} \text{has zeros at } q a = k \pi \Rightarrow \\
\frac{q}{\hbar} = k \frac{\pi}{a}, \quad k = 1, 2, 3, \ldots \\
(\text{zeros at } k = 0 \quad \sin \theta = 1) \end{cases} \]

The zeros appear at smaller \(q / \hbar\) than those for a uniformly charged sphere of the same radius.
The radius of the 1st Bohr orbit is \( A_1 = \frac{4\pi \xi E_0 h^2}{Ze^2 M} \), where \( M = \frac{m_1 m_2}{m_1 + m_2} \) is the effective mass.

a) Proton: \( m_p = 938.3 \text{ MeV}/c^2 \)

\[
M = \frac{105.66 \cdot 938.3}{105.66 + 938.3} = 94.37 \text{ MeV}/c^2 \Rightarrow A_1 = 284 \text{ fm}
\]

b) \(^{12}\) C nucleus: \( M = 105 \text{ MeV}/c^2 \) (Since the nucleus mass is 12\( \cdot m_p \) which is much larger than \( m_p \), the situation is the same as that of electron orbiting a proton: the central charge can be considered immobile).

\[
A_1 = \frac{4\pi \xi E_0 h^2 c^2}{Ze^2 (Mc^2)} = \left( \frac{4\pi \xi h c}{e^2} \right) \cdot \frac{h c}{Z \cdot M(\text{MeV})} = 137.1 \cdot 10^{-34} \text{ J} \cdot \text{s} \cdot \frac{3 \cdot 10^{-23} \text{ fm}}{S} \cdot \frac{1}{1.6 \cdot 10^{-13} \text{ MeV}} \cdot \frac{M(\text{MeV})}{J} = 137.1 \cdot 196.9 \text{ fm}
\]

\[
= 42.8 \text{ fm}
\]

c) Since \( A \cdot 938 \text{ MeV} \gg 105.66 \text{ MeV} \), then \( M = M\mu = 105.66 \text{ MeV} \);

The radius of the first orbit, \( A_1 = \frac{4}{Z} \frac{h^2}{m c^2} = \frac{137.196.9}{Z \cdot M(\text{MeV})} \text{ fm} = 3.113 \text{ fm} \); The radius of the n\(^{th}\) orbit is then \( A_n = n^2 \cdot A_1 = 3.113 n^2 \text{ fm} \). The radius of the nucleus is \( R = 1.2 \cdot A^{1/3} = 1.2 \cdot 208^{1/3} = 1.2 \cdot 5.825 = 7.110 \text{ fm} \).

Thus, the first orbit that lies outside the nucleus has \( h = 2 \) (\( A_2 = 4.3.113 \approx 12.4 \text{ fm} > R \)). The transition energy from \( n = 3 \) to \( n = 2 \) levels is \( \Delta E = E_3 - E_2 = \frac{M(\text{MeV}) \cdot 2^2}{2 \cdot 137^2} \left( \frac{1}{4} - \frac{1}{3^2} \right) \approx 2.63 \text{ MeV} \).
a) binding energy is \[ \frac{M Z^2 e^4}{2n^2 (4\pi \varepsilon_0 k)^2} = \frac{M (MeV) \cdot Z^2}{2n^2 \cdot 137^2} \]

In 2\(s\) state \(n=2\), and we have \(Z=6\), \(M = \frac{105.66 \cdot 12.938}{105.66 + 12.938} \approx 105 \text{ MeV} \Rightarrow E \approx \frac{105 \cdot 36}{8 \cdot 137^2} \approx 0.025 \text{ MeV} = 25 \text{ keV} \]

b) Assuming the model I,
\[ \rho(r) = \begin{cases} \rho_0 & r < a \\ 0 & r > a \end{cases} \]
we can calculate the difference between the potential of a point charge (as in part a of this problem) and potential of the homogeneous spherical distribution \(\rho(r)\). The latter potential is

\[ V(r) = \begin{cases} \frac{Z e^2}{4 \pi \varepsilon_0} \left[ \frac{r^2}{a^2} - 2 \right] & r \leq a \\ -\frac{Z e^2}{4 \pi \varepsilon_0 r} & r > a \end{cases} \]

Thus,
\[ \Delta V = V(r) - \left( -\frac{Z e^2}{4 \pi \varepsilon_0 r} \right) = \]
\[ = \frac{Z e^2}{4 \pi \varepsilon_0} \left[ \frac{r^2}{a^2} - 2 \right] + \frac{Z e^2}{4 \pi \varepsilon_0 r} = \frac{Z e^2}{4 \pi \varepsilon_0} \left[ \frac{r^2}{a^2} + \frac{a}{r} - 2 \right] \]

at \(r \leq a\), and \(\Delta V=0\) at \(r > a\).

The unperturbed wave function in 2\(s\) state is given by
\[ \psi_{2s} = \frac{2}{\sqrt{4\pi}} \left( \frac{2}{2a_B} \right)^{3/2} \left( 1 - \frac{2r}{a_B} \right) e^{-\frac{2r}{2a_B}} \]

\[ |\psi_{2s}|^2 = \frac{1}{\pi} \left( \frac{2}{2a_B} \right)^3 \left( 1 - \frac{2r}{a_B} \right)^2 e^{-\frac{2r}{a_B}} \]
Where $a_B$ is the Bohr radius. The correction from the first order perturbation theory is

$$\Delta E = \int_0^\infty \left| Y_{l=0}(r) \right|^2 \Delta V(r) \cdot 4\pi r^2 \, dr,$$

where we use spherical coordinates and immediately perform integration with respect to angles $\int d\phi \int d\cos \theta = 4\pi$, since the function depends only on $r$. Using the expression for $\Delta V(r)$ we obtain

$$\Delta E = 4\pi \cdot \frac{Ze^2}{4\pi \varepsilon_0 a} \cdot \frac{1}{\pi} \left( \frac{Z}{2a_B} \right)^3 \int_0^\infty (1 - \frac{Z}{a_B})^2 \cdot e^{-\frac{Z}{a_B}} \cdot \left( \frac{r^2}{a^2} + \frac{a^2}{r} - 2 \right) \cdot r^2 \, dr.$$

Introducing dimensionless variable $x = \frac{r}{a}$ we obtain:

$$\Delta E = \frac{4Z^4 e^2}{8 \cdot 4\pi \varepsilon_0 a} \left( \frac{a}{a_B} \right)^3 \int_0^1 (1 - \frac{Z}{a_B} x)^2 \cdot e^{-\frac{Z}{a_B} x} \cdot \left( x^2 + \frac{1}{x} - 2 \right) \cdot x^2 \, dx.$$

In our problem the Bohr radius $a_B = \frac{4\pi \varepsilon_0 \hbar^2}{e^2 M} = 258 \text{ fm}$

nucleus radius $a = 1.2 \cdot 1^{1/3} \approx 2.75 \text{ fm}.$

Now it is easy to estimate the integral: the term $\frac{Z}{a_B} x$ becomes $6 \cdot \frac{2.75}{258} \cdot x \approx 0.06 \cdot x$ which is very small compared to 1 even at the upper limit $x = 1$. Thus we can discard $\frac{Z}{a_B} x$ and the error introduced will be very small ($\Delta E \approx 6.4 \cdot 1.44 \text{ MeV.f.u.} \cdot \left( \frac{2.75}{258} \right)^3 \cdot 0.034 \approx 13.97 \text{ eV}$)

\[
\left\{ \frac{1}{5} + \frac{1}{2} - \frac{2}{3} \right\} \approx \frac{6.4 \cdot 1.44 \text{ MeV.f.u.} \cdot \left( \frac{2.75}{258} \right)^3 \cdot 0.034}{2 \cdot 2.75 \text{ fm}}.
\]