a) The total charge $Q = Z e$ in this case, radius $R = 1.24 \text{ A}^{1/3}$ fm.

The problem 1 of this assignment allows to write down the Coulomb term as $U = \frac{3Z^2e^2}{20\pi\varepsilon_0 \cdot 1.24 \text{ A}^{1/3}}$.

On the other hand, semi-empirical formula reads $U = \frac{Qe^2}{A^{1/3}}$, thus we can identify $A_C$ with

$$A_C = \frac{3e^2}{20\pi\varepsilon_0 \cdot 1.24 \text{ fm}} = \frac{3}{5} \cdot \frac{e^2}{4\pi\varepsilon_0 \cdot 1.24 \text{ fm}} = \frac{3}{5} \cdot \frac{1}{137} \cdot \frac{1.97 \text{ fm \cdot MeV}}{1.24 \text{ fm}}$$

$$= 0.695 \text{ MeV}$$

b) The binding energy of a nucleus $(Z, A)$ is $(A \text{ odd})$

$$B(Z, A) = A_N \cdot A - A_S A^{2/3} - A_C Z^2 A^{1/3} - A_A (A-2Z)^2 / A$$

$$A_N = 15.56 \text{ MeV}, \quad A_S = 17.23 \text{ MeV}, \quad A_A = 23.285 \text{ MeV} \Rightarrow$$

With $B = 1457 \text{ MeV}$ this results in $A_C \approx 0.694 \text{ MeV}$

c) $^{235}_{92} \text{U} \rightarrow ^{35}_{37} \text{Br} + ^{57}_{47} \text{La} + 3n$

The binding energy of $^{37}_{35} \text{Br}$ is $B(^{35}_{37} \text{Br}) = A_N (87) - A_S (87)^{2/3} - A_C (35^2 (87)^{1/3} - A_A (87-70)^2 / 87)$; similarly for other species,

and the released energy is the difference between the energies of "incoming" and "emerging" species:

$$\Delta E = B(^{35}_{87} \text{Br}) + B(^{57}_{145}) - B(^{92}_{235}) \quad [\text{note that total number of protons and neutrons is the same on both sides in this decay process, so the mass terms } ZM_p \text{ and } (A-Z)M_n \text{ compensate each other}].$$

Calculation gives $\Delta E \approx 154 \text{ MeV}$. 

3. Atomic rest mass energy (including electrons) is

\[ M(Z, A)C^2 = 2M_n C^2 + (A - Z)M_p C^2 - B(Z, A), \]

where

\[ M_n C^2 = 938.791 \text{ MeV} \] is the mass of a hydrogen atom (proton and electron); \[ M_p C^2 = 938.573 \text{ MeV} \]

\[ B(Z, A) = \alpha_n A - \alpha_p A^{2/3} - \alpha_c \frac{Z^2}{A^{1/3}} - \alpha_A \left(\frac{A - 2Z}{A}\right)^2 + \delta_a \]

for \( ^{40}_{19}K \):

\[ M(Z, A)C^2 = 19 \times 938.791 + 21 \times 938.573 - 15.56 \times 40 + \\
+ 17.23 \times (40)^{2/3} + 0.697 \frac{19^2}{(40)^{1/3}} + 23.285 \times \frac{A}{40} + 12.0 \times \frac{12}{40^{1/2}} = \\
37224.984 \text{ MeV} \]

for \( ^{40}_{20}Ca \) the mass can be easily obtained from that of \( ^{40}_{19}K \) by removing one \( M_n C^2 \), adding one \( M_p C^2 \) and correcting for \( \alpha_c, \alpha_A \) and \( \alpha_p \) terms (all others depend only on \( A \) and stay unchanged):

\[ M(20, 40) = M(19, 40) - M_n C^2 + M_p C^2 - \alpha_c \frac{19^2}{40^{1/3}} - \alpha_A \frac{20^2}{40^{1/3}} - \\
- \alpha_A \frac{1}{40} \times 2 \times \frac{12.0}{40^{1/2}} = M(18, 40) + M_p C^2 - M_n C^2 + \\
- \alpha_A \frac{24.0}{40^{1/2}} + \alpha_c \frac{(20^2 - 19^2)}{40^{1/3}} = 37226.028 \text{ MeV} \]

Similarly \( M(18, 40) = 37221.416 \text{ MeV} \) Thus the energy diagram obtained with the help of the semi-empirical mass formula would look like \( ^{40}_{19}K \). To understand this, look at Williams, Fig. 4.6. There is a considerable discrepancy.
between the formula and experiment in the vicinity of $A=40$.
(up to $0.1 \text{meV per nucleon}$, or $\sim 1 \text{meV per nucleon}$!)

The origin of the dotted line in Fig. 58:

$\beta^+$ decay is a decay of type

$$(Z,A) \rightarrow (Z-1,A) + e^+ + \nu_e$$

The mass of original atom is $M(Z,A)$; the mass of products is $M(Z-1,A) + m_e$. Note however that in $\beta^+$ decay only nuclear mass changes (plus emitted $e^+\nu_e$ and $\nu_e$ energies of course), while $M(Z)$ and $M(Z-1)$ have different number of electrons counted in. In terms of nuclear masses the conservation of energy requires

$$M(Z,A) > M(Z-1,A) + m_e$$

(plus surplus of the energy to be shared between kinetic energies of $e^+$ and $\nu_e$). Now adding $2m_e$ to both sides we obtain:

$$M(Z,A) > \left[ M(Z-1,A) + (Z-1)m_e \right] + m_e + m_e = M(Z-1,A) + M(Z-1,A) + 2m_e$$

Thus $\beta^+$ decay is allowed only if the parent state is more than $2m_e$ above the daughter one. (in atomic masses). The remaining gap from the dashed line and the daughter state is the energy released in the form of kinetic energy of $e^+$ and $\nu_e$. 


Problem 3

From the LBNL-LUND University summary drawings for A = 40 we can get the figure by converting amu to MeV.

From the chart of nuclides tool, we can get the mass defects $\Delta$

<table>
<thead>
<tr>
<th>Nuclide</th>
<th>$\Delta$ (MeV)</th>
<th>$\Delta$ (MeV) - $\Delta$(40Ar)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{40}$Sc</td>
<td>-20.5 MeV</td>
<td>14.5</td>
</tr>
<tr>
<td>$^{40}$Ca</td>
<td>-34.8 MeV</td>
<td>0.2</td>
</tr>
<tr>
<td>$^{40}$K</td>
<td>-33.5 MeV</td>
<td>1.5</td>
</tr>
<tr>
<td>$^{40}$Ar</td>
<td>-35.0 MeV</td>
<td>0.0</td>
</tr>
<tr>
<td>$^{40}$Cl</td>
<td>-27.6 MeV</td>
<td>7.4</td>
</tr>
<tr>
<td>$^{40}$S</td>
<td>-22.9 MeV</td>
<td>12.1</td>
</tr>
<tr>
<td>$^{40}$P</td>
<td>-8.0 MeV</td>
<td>27.0</td>
</tr>
</tbody>
</table>

$\Delta = (M_{amu}) - A$ amu

$\Delta$ (MeV) = $M$(MeV) - $M$(40Ar, MeV)
Problem 4. (half-lives are used!)

\[ ^{238} \text{U} \]
\[ \rightarrow 1.39 \times 10^5 \text{y} \]

\[ ^{235} \text{U} \]
\[ \rightarrow 4.57 \times 10^9 \text{y} \]

\[ ^{234} \text{Th} \]
\[ \rightarrow 24 \text{d} \]
\[ \rightarrow ^{234} \text{Pa} \]
\[ \rightarrow 6.66 \text{h} \]
\[ \rightarrow ^{234} \text{U} \]
\[ \rightarrow 2.48 \times 10^5 \text{y} \]

The longest living isotopes in

\[ A_{\text{mod}} = 0, 1, 2, 3 \] series are:

(see atomic table)

\[ ^{232} \text{Th} \]
\[ \rightarrow \tau_1 = 1.39 \times 10^5 \text{y} \]

\[ ^{237} \text{Np} \]
\[ \rightarrow \tau_2 = 2.2 \times 10^6 \text{y} \]

\[ ^{238} \text{U} \]
\[ \rightarrow \tau_3 = 4.57 \times 10^9 \text{y} \]

\[ ^{235} \text{U} \]
\[ \rightarrow \tau_4 = 7.13 \times 10^8 \text{y} \]

The relative abundance of these elements would be

\[ e^{-\frac{t}{\tau_1}} : e^{-\frac{t}{\tau_2}} : e^{-\frac{t}{\tau_3}} : e^{-\frac{t}{\tau_4}} = 1 : 0 : 0.34 : 10^{-5} \]

Note however, that we assumed here that those long-living elements are "bottlenecks" for any decay series (A_{\text{mod}} = 0, 1, 2, 3 respectively). However, as we have seen, for A even the decay series can branch and...
Problem 5

Since both protons and neutrons are fermions, they should conform to the Pauli principle: two identical particles cannot be in the same quantum state. If we use momentum \( p \) to label the quantum states, the Pauli principle tells us that only two identical particles (spins up and down, which makes their quantum states different!) can have some fixed value of \( p \).

Since the system tries to minimize its energy, non-interacting particles (kinetic energy only) try to make their momenta as small as possible. Being unable to have the same momentum \( p = 0 \), they take over all possible momenta from 0 to some maximal one, \( p_F \) (Fermi momentum). In finite volume, momentum is quantized, the only allowed values are \( p = \left( \frac{2\pi n_x}{L_x}, \frac{2\pi n_y}{L_y}, \frac{2\pi n_z}{L_z} \right) \), where \( n_x, n_y, n_z \) are integers and \( L_x, L_y, L_z \) are the sides of the box \( \Rightarrow \) the volume in the momentum space occupied by a single state is

\[
\mathcal{V} = \left( \frac{2\pi \hbar}{L_x} \right)^3 = \left( \frac{2\pi \hbar}{L_y} \right)^3 \cdot \left( \frac{2\pi \hbar}{L_z} \right)^3 = \left( \frac{2\pi \hbar}{L} \right)^3,
\]

hence the Fermi sphere with \( \frac{4\pi}{3} \) identical particles \( p_F \) can accommodate \( 2 \cdot \frac{4\pi}{3} p_F^3 V \) \( \frac{4\pi}{3} p_F^3 \) \( \mathcal{V} \), where \( \frac{4\pi}{3} p_F^3 \) is the volume of the sphere and factor 2 accounts for the fact that two particles (spin up and down) can occupy the same \( p \)-state. Since protons and neutrons are not identical, the above expression applies separately to each of them. If their number is the same, the proton and neutron Fermi momenta are the same as well.
and the total number of nucleons is \( 4 \cdot \frac{4\pi}{3} \rho^2 V \) 

\[ \frac{16\pi}{3} \rho^3 \frac{V}{(2\pi\hbar)^3} = A \Rightarrow \rho^3 = \frac{3}{16\pi} \frac{A (2\pi\hbar)^3}{V} \]

The Fermi energy is \( E_F = \frac{\rho^2}{2M} \), where \( M \) is the nucleon mass (here we assume \( m_p = m_n \)); using \( V = \frac{4\pi}{3} R_0^3 A \) for the volume of the nucleus we obtain

\[ \rho_F = \left[ \frac{3}{16\pi} \cdot A \frac{(2\pi\hbar)^3}{4\pi/3 R_0^3 A} \right]^{1/3} = \frac{2\pi\hbar}{R_0} \left[ \frac{8}{6A\pi^2} \right]^{1/3}, \text{ and hence} \]

\[ E_F = \frac{\hbar^2}{2MR_0^2} \left[ \frac{8}{6A\pi^2} \right]^{3/2} = \frac{\hbar^2}{2MR_0^2} \left[ \frac{(2\pi)^3 \cdot 8}{6A\pi^2} \right]^{2/3} = \frac{\hbar^2}{2MR_0^2} \left[ \frac{9\pi}{8} \right]^{2/3} \]

The number of states in the range \( p, p + dp \) is

\[ dN = \frac{4\pi p^2 dp \cdot V}{(2\pi\hbar)^3} \], \text{ in each state we can have 4 nucleons (spin up or down, proton or neutron), and each of them has energy \( \frac{p^2}{2M} \) → the total kinetic energy is} \]

\[ E = \int_0^{\rho_F} \frac{4p^2}{2M} dN = \int_0^{\rho_F} \frac{4p^2}{2M} \cdot \frac{4\pi p^2 V}{(2\pi\hbar)^3} dp = \frac{4}{15\pi} \left( \frac{\hbar c}{R_0} \right)^2 \frac{1}{Mc^2} \left( \frac{9\pi}{8} \right)^{3/2} A \]

For \( {}^{16}O \), \( A = 16, M^2 = 938 \text{ MeV}, R_0 = 1.2 \text{ fm} \Rightarrow E = 320 \text{ MeV} \)
The level spacing at Fermi energy

\[ \Delta E = \left( \frac{dn}{dE} \bigg|_{E=E_F} \right)^{-1} = \frac{4\pi p^2 v dp}{(2\pi \hbar)^3 dE} \]

Since \( \frac{p^2}{2M} = E \), then \( \frac{p dp}{M} = dE \Rightarrow \frac{dp}{dE} = \frac{M}{p} \); thus

\[ \Delta E = \frac{(2\pi \hbar)^3}{4\pi p_F^2 M} \]

If in a nucleus one proton is transformed into a neutron, it has to be moved up in energy (if in initial state \( N \geq 2 \)):

protons \hspace{1cm} \text{neutrons} \hspace{1cm} 3 \Delta E

At each step the required energy to transform \( Z, N \) into \( Z-1, N-1 \) is

\[ \frac{N-2}{2} \Delta E \text{ (factor of } \frac{1}{2} \text{ because one level can accommodate two neutrons).} \]

To sum up all these energies, required to pass through the sequence \( Z=\frac{A}{2}, N=\frac{A}{2} \Rightarrow A+1, \frac{A}{2}, A-2, \frac{A}{2}, A+2 \Rightarrow \ldots \Rightarrow Z, Z \), it is convenient to approximate the sum by an integral (since at each step the required energy is small). Writing down

\[ \frac{dE}{dz} \approx \frac{N-2}{2} \Delta E \]

we obtain:

\[ E_A = \int \frac{dE}{dA} \, dz = \]

\[ = \int \frac{N-2}{2} \Delta E \, dz = \left( A - 2z \right)^2 \Delta E \] \hspace{1cm} \text{since the asymmetry term is } A \left( A - 2z \right)^2 \hspace{1cm} \text{we obtain } \]

\[ c_A = A \frac{\Delta E}{\Delta E} = \]

\[ = \frac{A}{8} \cdot \frac{(2\pi \hbar)^3}{4\pi} \left( \frac{64\pi^2}{9} \right)^{1/8} \cdot \frac{3}{16} \left( \frac{8\pi^2}{9} \right)^{1/8} \frac{1}{R^2 M} \approx 15 \text{ MeV} \]