The decay rate of a radioactive isotope is \( \frac{dN}{dt} = \frac{1}{\tau} N \) where \( \tau \) is the mean life and \( N \) is the number of atoms. If this isotope is a product of decay of another (parent) isotope, then the number of atoms of the daughter isotope must have income of \( \frac{1}{\tau} \bar{N} \) from parent. (\( \tau, \bar{N} \) are mean life and number of atoms of parent). Clearly, if \( N \) is too small, the depletion \( \frac{1}{\tau} N \) is smaller than income and \( N \) increases; if \( N \) is too large, the depletion is larger than income, \( \frac{1}{\tau} N > \frac{1}{\tau} \bar{N} \) and \( N \) decreases. Thus after time long enough (which has passed entirely), the intermediate states are in dynamic equilibrium with parents:

\[
\frac{1}{\tau} N = \frac{1}{\tau} \bar{N} \implies N = \frac{\bar{N}}{\tau}. 
\]

In our case we have the reaction chain:

\[
\begin{align*}
N^{238}(U) & \quad \tau_1 = 6.5 	imes 10^{9} \  y \\
\downarrow & \\
N^{234}(U) & \quad \tau_2 = 3.53 \times 10^{7} \  y \\
\downarrow & \\
N^{230}(Th) & \quad \tau_3 = 1.12 \times 10^{5} \  y \\
\downarrow & \\
N^{226}(Ra) & \quad \tau_4 = 2.31 \times 10^{3} \  y
\end{align*}
\]

The dynamic equilibrium conditions are: (If the factors become):

\[
\frac{1}{\tau_1} N^{238}(U) = \frac{1}{\tau_2} N^{234}(U) = \frac{1}{\tau_3} N^{230}(Th) = \frac{1}{\tau_4} N^{226}(Ra).
\]
so that \[ N^{(226\text{Ra})} = \frac{T_4}{T_2} N^{(238\text{U})}. \]

The weight of \( ^{238}\text{U} \) is 0.15\%, \( \Rightarrow \) hence \[ M^{(238\text{U})} = 0.15\% \times 1.5 \text{ kg} = 0.15 \times 10^{-2} \times 1.5 \text{ kg} = 0.15 \times 10^{-2} \text{ kg} = 1.5 \text{ kg}, \]
or the number of atoms is

\[ N^{(238\text{U})} = \frac{1.5 \text{ kg}}{1.5 \text{ kg} / \text{mol}} \quad \text{ where } A \text{ is the atomic mass unit}. \]

Denoting the mass of Ra by \( M \) we obtain:

\[ N^{(226\text{Ra})} = \frac{M}{226 \text{ A}} \Rightarrow \frac{M}{226 \text{ A}} = \frac{T_4}{T_2} \times \frac{1.5 \text{ kg}}{238 \text{ A}} \Rightarrow \]

\[ M^{(226\text{Ra})} = \frac{T_4}{T_2} \times \frac{226 \text{ A} / 238 \text{ A} \times 1.5 \text{ kg} = 2.31 \times 10^3 \text{ kg}}{6.45 \times 10^4} = 2.26 \times 1.5 \text{ kg} = 0.34 \times 10^{-6} \text{ kg} = 0.51 \times 10^{-6} \text{ kg} = 0.51 \text{ mg} \]

[for those who tried to use the answer in Williams, is a reference: there is a misprint in the book, the correct answer is 0.34 mg].

Now, we know the decay rate of each isotope in the sequence (and those rates are the same, \( \frac{1}{T_4} N^{(238\text{U})} \)), since we are in dynamical equilibrium. Each decay of each isotope in a sequence produces an d-particle, from the table you can see that there are total 8 d-decays occurring on the way to a stable isotope \( \Rightarrow \) the half rate of production of helium (if particles) is \( \frac{1}{2} N^{(238\text{U})} = \)

\[ \frac{8 \times 1.5 \text{ kg}}{T_4} = \frac{8 \times 1.5 \text{ kg}}{238 \text{ A}} \]

for helium: \[ \frac{\text{Imass}}{A \times A} = \frac{8 \times 1.5 \text{ kg}}{T_4 \times 238 \text{ A}} \Rightarrow \frac{\text{Imass}}{A \times A} = \frac{3.2}{238 \times 1.5 \text{ kg} \times \frac{1}{T_4}} \approx 3.13 \times 10^7. \]
In the lab frame, the projectile has velocity $V_1$ and target is at rest. Let us consider a frame which moves with velocity $U$ with respect to the lab. Clearly, when measured in this moving frame, the velocity of the target is $-U$, and that of the projectile is $V_1 - U$ (since the velocity addition is of vector nature, we will need this in the next problem, here we just assume the direction of $x$-axis to be along $V_1$).

Thus, in the moving frame, the momentum of the target is $-M_2 U$, the momentum of the projectile is $M_1 (V_1 - U)$. In the c.m. frame, these momenta have to be equal and opposite, such that their sum is zero. This condition results in

$$M_1 (V_1 - U) = M_2 U \Rightarrow M_1 V_1 = (M_2 + M_1) U \Rightarrow$$

$$U = \frac{M_1 V_1}{M_1 + M_2}$$

the momenta in the c.m. system are

consequently $P_c = M_1 (V_1 - U) - M_2 U = \frac{M_1 M_2 V_1}{M_1 + M_2} = \frac{P_1 M_2}{M_1 + M_2}$

(since $P_1 = M_1 V_1$). Kinetic energy in the c.m. frame is

$$T_c = \frac{P_c^2}{2M_1} + \frac{P_c^2}{2M_2} = \frac{P_c^2}{2} \left( \frac{M_1 + M_2}{M_1 M_2} \right)$$

or using

the expression for $P_c$:

$$T_c = \frac{P_c^2 M_2^2}{2(M_1 + M_2)} \cdot \frac{M_1 + M_2}{M_1 M_2} = \frac{T_1 M_2}{M_1 + M_2}$$

It is instructive to check that the rest of the projectile kinetic
energy is hidden in the energy of the center of mass motion.

\[
T_{c.m.} = \frac{(M_1 + M_2) U^2}{2} = \frac{(M_1 + M_2) M_3^2 v_1^2}{2 (M_1 + M_2)^2} = \frac{M_1 T_4}{M_1 + M_2},
\]

so that naturally

\[
T_c + T_{c.m.} = T_4
\]

**Williams 7.2.**

The kinetic energy of products in the c.m. frame is

\[
\frac{(P_3')^2}{2 M_3} + \frac{(P_4')^2}{2 M_4} = T_c' \Rightarrow
\]

\[
\left( \frac{P_3'}{\sqrt{2}} \right) \left( \frac{1}{M_3} + \frac{1}{M_4} \right) = T_c'
\]

If the velocity in c.m. is \( \vec{V} \), then, transforming back to the lab, the velocity in the lab frame becomes \( \vec{V} = \vec{V} + \vec{U} \) (note that in the geometry shown in the picture, this affects only the \( x \)-component, while the \( y \)-component stays unchanged!). Thus, for momenta in the lab frame we obtain:

\[
\begin{align*}
P_x &= P_i \cos \Theta_i = P_3' \cos \Theta + M_3 U_x, \\
P_y &= P_i \sin \Theta_i = P_3' \sin \Theta
\end{align*}
\]

where \( U \) is the c.m. velocity calculated in the previous problem.
1) $d + d \rightarrow n + ^3_2\text{He}$ (d is $^1_1\text{H}$)
2) $d + t \rightarrow n + d$ (t is $^3_1\text{H}$, d is $^2_2\text{He}$)

Reactions: In both reactions, the number of protons and neutrons is unchanged, hence, the energy comes only from the mass defect. Writing down the mass defects on the left and right sides of the reaction schemes we obtain

**Reaction 1**:

\[
2 \cdot 0.014102\text{u} \rightarrow 0.008665\text{u} + 0.016030\text{u}
\]

\[
0.028204 \rightarrow 0.024635
\]

\[
W = 0.028204 - 0.024635 = 0.003569\text{u} = 3.2626\text{ MeV} \frac{\text{u}}{\text{c}^2}
\]

**Reaction 2**:

\[
0.014102 + 0.01605U \rightarrow 0.008665 + 0.002603
\]

\[
0.030152 \rightarrow 0.011268
\]

\[
W = 0.030152 - 0.011268 = 0.018884\text{u} \times 13.540\text{ MeV} \frac{\text{u}}{\text{c}^2}
\]

W-value is positive in both cases, so that the reactions are **exothermic**, the energy is **released**.

**B)** Let us now calculate kinetic energy. We will follow Williams' notations, with $d$ being the projectile, $d$ being the target.

In the products, let us denote neutron by $n$ and remaining particles by $x$. 


The energy of projectile (lab) \( E_1 = \frac{M_1 V_1^2}{2} \) \( \Rightarrow \) \( V_1 = \sqrt{\frac{2E_1}{M_1}} \)

In the c.m. frame (see problem 7.1) the total kinetic energy is

\[
T_c = \frac{T_1 M_2}{M_1 + M_2}
\]

After the reaction \( T_c' = T_c + \Delta = \frac{T_1 M_2}{M_1 + M_2} + \Delta \).

and this is the energy totally available for build up of kinetic energy of products (the neither momentum, nor kinetic energy of the center of mass can not be changed, regardless of the internal motion of constituent particles). Since in c.m. from both products have the same momentum \( P_2' \), it is easy to obtain

\[
\frac{(P_2')^2}{2} \frac{M_3 + M_4}{M_2 M_4} = T_c' \Rightarrow P_c' = \sqrt{\frac{2T_c' M_3 M_4}{M_2 M_4}} = \sqrt{\frac{2(T_c' + \Delta) M_3 M_4}{M_2 M_4}}
\]

Using notations of William 7.2: the components of the momentum of neutron in the lab frame are

\[
P_x = P_c' \cos \theta + M_3 U, \quad P_y = P_c' \sin \theta
\]

its kinetic energy is equal then to

\[
T = \frac{P_x^2 + P_y^2}{2M_3} = \left( \frac{P_c^2}{2} \right)^2 \frac{M_3 U^2}{2} + \frac{M_3 U^2}{2} + P_c' U \cos \theta
\]

in the c.m. frame angle \( \theta \) can have any value. The expression for \( T \) has maximum at \( \cos \theta = 1 \) (i.e. \( \theta = 0 \)) which is obvious: it is favorable to have the product moving forward in c.m. frame, then it's energy in lab frame is maximal.

Thus, we obtain

\[
T_{\text{max}} = \left( \frac{P_c}{2} \right)^2 + \frac{M_3 U^2}{2} + P_c' U = \left( \frac{P_c + M_3 U}{2} \right)^2
\]
Using \( U = \frac{M_1 V_1}{M_1 + M_2} \) (see Williams 7.4) we finally arrive at:

\[
T_{\text{max}} = \frac{\Delta}{2M_0} \left[ \sqrt{2M_2 M_4 \left( \frac{T_1 M_2}{M_2 + M_4} + \alpha \right)} + M_3 \frac{M_1 V_1}{M_1 + M_4} \right]^2 = \]

\[
= \frac{1}{2M_0} \left[ \sqrt{2M_3 M_4 \left( \frac{T_1 M_2}{M_2 + M_4} + \alpha \right)} + \frac{M_3}{M_1 + M_4} \sqrt{2T_1 M_4} \right]^2
\]

It is easy to see that the atomic unit \( 931.5 \text{ MeV} \) is not needed here at all, still let’s demonstrate it once: denoting \( Q = 931.5 \text{ MeV} \) we have for

**Reaction 1** : \( M_4 = M_2 = 2a \), \( M_3 = a \), \( M_1 = 3a \) and \( Q = 3.27 \text{ MeV} \)

(from part A1) \( \Rightarrow \)

\[
T_{\text{max}} = \frac{4}{2a} \left[ \sqrt{\frac{\alpha^2}{4a} \left( \frac{4M_1 V_1}{2a} + 3.27 \text{ MeV} \right)} + \frac{a}{4a} \sqrt{8M_1 V_1 \cdot 2a} \right]^2 =
\]

\[
= \frac{1}{2a} \left[ \sqrt{\frac{\alpha}{2} \left( 2 \text{ MeV} + 3.27 \text{ MeV} \right)} + \sqrt{1 \text{ MeV} \cdot \alpha} \right]^2 =
\]

\[
= \frac{1}{a} \left[ \sqrt{1.5 \cdot 5.27 \text{ MeV} \sqrt{1 \text{ MeV} \cdot \alpha}} \right]^2 \text{ MeV} = \frac{1}{2} \left[ \sqrt{1.904} \right]^2 \text{ MeV} =
\]

\[
= \frac{1}{2} \left[ 2.812 + 1 \right]^2 \text{ MeV} = 7.26 \text{ MeV}
\]

In the same way (omitting “\( a \)” from the very beginning):

**Reaction 2** : \( M_4 = 2 \), \( M_2 = 3 \), \( M_3 = 4 \), \( M_1 = 4 \), \( Q = 17.5 \text{ MeV} \)

\[
T_{\text{max}} = \frac{1}{2} \left[ \sqrt{\frac{2}{3} \left( \frac{4M_1 V_1}{2a} + 17.513 \right)} + \frac{1}{3} \sqrt{8M_1 V_1 \cdot 2a} \right]^2 =
\]

\[
= \frac{1}{2} \left[ \sqrt{\frac{2}{3} \cdot 17.513 + 17.513} + \frac{4}{3} \right]^2 \text{ MeV} = 80.1 \text{ MeV}
\]