1. (10 pts.) Electrostatic Potential and Electrostatic Force.

A thin rod of length $D$ carries a total charge of $Q$ distributed uniformly over its length. One end of the rod is fixed at the origin, the other end fixed at a distance $D$ along the negative $x$–axis, as shown in the figure below.

(a) By integrating over the length of the rod, find a symbolic expression for the electrical potential $V(x, 0, 0)$ everywhere along the positive $x$–axis (assume $V \to 0$ at $\infty$).

(b) You are given $Q = 50.0 \, \mu\text{C}$, and $D = 20.0 \, \text{cm}$. Calculate the force $\vec{F}$ (in newtons) exerted by the rod on a test charge $q = 2.0 \, \mu\text{C}$ located at $(x_0, 0, 0)$ where $x_0 = 1.50 \, \text{m}$. 
\[ V(x,0,0) = \frac{1}{4\pi \varepsilon_0} \int \frac{dQ}{r} \quad \text{and} \quad V(x,0,0) = \frac{1}{4\pi \varepsilon_0} \int \frac{dQ}{|\mathbf{r} - \mathbf{r}'|} \]

\[ r = x - x' \quad \quad |\mathbf{r} - \mathbf{r}'| = |x - x'| = (x - x')^1 \]

\[ dQ = \lambda \, dx' \quad \lambda = \text{linear charge density} \]

\[ \frac{dQ}{D} \, dx' \]

\[ V(x,0,0) = \frac{1}{4\pi \varepsilon_0} \int_{-D}^{0} \frac{Q/\lambda \, dx'}{(x-x')} \]

Variable substitution: let \( u = x-x' \)

\[ dx' = -du \]

\[ x' = -D \rightarrow u = x + D \]

\[ x' = 0 \rightarrow u = x \]

\[ V(x,0,0) = \frac{-Q}{4\pi \varepsilon_0 \, D} \int_{x+D}^{x} \frac{du}{u} \]

\[ = \frac{-Q}{4\pi \varepsilon_0 \, D} \ln u \bigg|_{x+D}^{x} \]

\[ = \frac{-Q}{4\pi \varepsilon_0 \, D} \ln \left( \frac{x}{x+D} \right) \]

\[ \text{(Note: } -\ln y = \ln \left( \frac{1}{y} \right) \text{)} \]

\[ V(x,0,0) = \frac{Q}{4\pi \varepsilon_0 \, D} \ln \left( \frac{x+D}{x} \right) \]
1(b) The easy way to do this is:

\[ E_x = -\frac{Q}{2\pi \epsilon_0 D} \quad E_y = -\frac{Q}{2\pi \epsilon_0 D} \quad E_z = -\frac{Q}{2\pi \epsilon_0 D} \]

In this case, it is clear by symmetry that

\[ E_y = E_z = 0 \quad \text{i.e.,} \quad \vec{E}(x,0,0) = E_x(x) \hat{i} \]

\[ E_x = -\frac{Q}{2\pi \epsilon_0 D} \cdot \frac{d}{dx} \left[ \ln \left( \frac{x+D}{x} \right) \right] \]

\[ = -\frac{Q}{4\pi \epsilon_0 D} \cdot \frac{x}{x+D} \cdot \left[ \frac{1}{x} - \frac{(x+D)}{x^2} \right] \]

\[ = -\frac{Q}{4\pi \epsilon_0 D} \cdot \frac{x}{x+D} \cdot \frac{x-x-D}{x^2} \]

\[ = \frac{Q}{4\pi \epsilon_0 D} \cdot \frac{x}{x+D} \cdot \frac{D}{x^2} = \frac{Q}{4\pi \epsilon_0 D(x+D)} \cdot \frac{D}{x^2} \]

\[ \vec{E}(x,0,0) = \frac{Q}{4\pi \epsilon_0 D(x+D)} \hat{i} \]

\[ \vec{F} = q \vec{E}(\vec{r}_0) \quad \vec{r}_0 = (x_0,0,0) \]

\[ \vec{F} = \frac{Qq}{4\pi \epsilon_0 x_0(x_0+D)} \hat{i} \]

\[ F = \frac{5 \times 10^{-5} \text{ C} \cdot 2 \times 10^{-12} \text{ C}}{4\pi (8.85 \times 10^{-12} \text{ Nm}^2/\text{C}^2)(1.50 \text{ m})(1.50 \text{ m} + 0.200 \text{ m})} \]

\[ = 0.35262 \text{ N} \]

\[ \vec{F} = 0.352 \hat{i} \text{ N} \]

or \[ F = 0.353 \text{ N in the } +X \text{ direction} \]
(b) Alternately: \[ F = \mathcal{E}(x,0,0) \]

\[ \mathcal{E}(x,0,0) = \frac{1}{4\pi\varepsilon_0} \int \frac{dQ\hat{\lambda}}{\frac{1}{2}r^2} \]

Here \( \hat{\lambda} = \lambda \), \( r^2 = (x-x')^2 \)

\[ dQ = \lambda \, dx' = \frac{Q}{D} \, dx' \]

\[ \mathcal{E}(x,0,0) = \frac{Q \hat{\lambda}}{4\pi\varepsilon_0 D} \int_{-D}^{D} \frac{dx'}{(x-x')^2} \]

Let \( u = x-x' \), \( du = -dx' \)

\( x' = -D \rightarrow u = x+D \)
\( x' = 0 \rightarrow u = x \)

\[ \mathcal{E}(x,0,0) = \frac{-Q}{4\pi\varepsilon_0 D} \int_{x+D}^{x} \frac{du}{u^2} \]

\[ = \frac{-Q}{4\pi\varepsilon_0 D} \left. \frac{1}{u} \right|_{x+D}^{x} \]

\[ = \frac{Q}{4\pi\varepsilon_0 D} \left[ \frac{1}{x} - \frac{1}{x+D} \right] \]

\[ = \frac{Q}{4\pi\varepsilon_0 D} \frac{x+D - x}{x(x+D)} \]

\[ = \frac{Q}{4\pi\varepsilon_0 D} \frac{x}{x(x+D)} \]

\[ F = \frac{Q}{4\pi\varepsilon_0 D} \frac{x}{x(x+D)} \]

\( \rightarrow \) same answer
2. (10 pts.) Gauss’ Law and Capacitance

In the figure below a capacitor is shown made from a solid conducting cylindrical rod of radius \( a \), and length \( L = 5.00 \text{ m} \), enclosed by a thin conducting cylindrical shell of the same length, but of radius \( b = 4.00 \text{ cm} \) \((b > a)\). A positive charge \( Q = 3.00 \text{ nC} \) is placed on the rod, and an equal but opposite charge of \(-Q\) on the cylindrical shell. A voltmeter connected between the rod and the shell reads \( \Delta V = 15.0 \text{ V} \).

For this problem you may neglect the edge/end effect on the capacitor.

(a) Using Gauss’s Law, find a symbolic expression for the electric field \( \vec{E} \) in the region \( a < r < b \), where \( r \) is the perpendicular distance from the axis of the cylinder.

(b) By explicit line/path-integration from the shell to the rod along a radial path, find a symbolic expression for the potential difference \( \Delta V \) from the shell to the rod.

(c) Find a symbolic expression for the capacitance \( C \) of this capacitor.

(d) Find the radius \( a \) of the inner cylindrical rod in centimeters.
(2)(a) Gauss' Law:

\[ \Phi_E = \oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{enc}}{\varepsilon_0} \]

Gaussian Surface

We pick as our Gaussian surface a cylinder (including endcaps) of radius \( \mathbf{r} \), length \( L < L \) (contained within the length of the cable).

By symmetry: \( \mathbf{E}(\mathbf{r}) = E(r) \mathbf{\hat{r}} \) (neglecting edge effect)

\( \Rightarrow \) the endcaps do not contribute to \( \Phi_E \)

\[ \Phi_E = 2\pi r L \mathbf{\hat{r}} \cdot E(r) \mathbf{\hat{r}} = 2\pi r L E(r) \]

for \( a < r < b \) \( \Rightarrow \) \( Q_{enc} = \frac{Q}{L} \)

\( \Rightarrow \) \( 2\pi r L E(r) = \frac{Q}{L} \varepsilon_0 \)

\[ E(r) = \frac{Q/L}{2\pi \varepsilon_0 r} = \frac{Q}{2\pi \varepsilon_0 L r} \]

\[ \mathbf{E}(r) = \frac{Q r}{2\pi \varepsilon_0 L r} \mathbf{\hat{r}} \]

for \( a < r < b \)

\[ = \frac{2keQr}{Lr} \quad (\frac{1}{4\pi \varepsilon_0} = ke) \]
2(b) Taking a radial path:  
\[ \Delta V = -\int E \cdot ds^2 = -\int_{r=a}^{r=b} \frac{Qr}{2\pi \varepsilon_0 L} \cdot \frac{r}{ds^2} \]

Shell \[ \Delta V = -\frac{Q}{2\pi \varepsilon_0 L} \ln \left( \frac{b}{a} \right) = \frac{2kQ}{L} \ln \left( \frac{b}{a} \right) \]

(c) \[ C = \frac{Q}{\Delta V} = \frac{2\pi \varepsilon_0 L}{\ln \left( \frac{b}{a} \right)} = \frac{L}{2k_e \ln \left( \frac{b}{a} \right)} \]

(d) we have \[ L = 5.00 \text{ m} \quad b = 0.0400 \text{ m} \]
\[ Q = 3.00 \times 10^{-9} \text{ C} \]
\[ \Delta V = 15.0 \text{ V} \]
\[ C = \frac{Q}{\Delta V} = \frac{3.00 \times 10^{-9} \text{ C}}{15.0 \text{ V}} = 2.0 \times 10^{-10} \text{ F} \quad (= 200 \text{ pF}) \]

\[ \ln \left( \frac{b}{a} \right) = \frac{2\pi \varepsilon_0 L}{C} = \frac{2\pi \left( 8.85 \times 10^{-12} \frac{C^2}{N \cdot m^2} \right)(5.00 \text{ m})}{2.0 \times 10^{-10} \frac{C^2}{N \cdot m}} \]

\[ = 1.390155... \]

\[ \frac{b}{a} = e^{1.390155} = 4.0155 \]

\[ a = \frac{b}{4.0155} = \frac{4.00 \text{ cm}}{4.0155} = 0.996 \text{ cm} \]

\[ a = 0.996 \text{ cm} \]
3. (10 pts.) Magnetic Field using Biot-Savart Law

In the figure below, a short segment of a thin wire of is bent into the shape of a semi-circle of radius \( R \) that lies in positive \( x \) half of the \( xy \)-plane, and is centered at the origin. It carries a current of \( I_0 \) in counter-clockwise direction relative to the \( +z \)-axis. (The “return current” is shielded from the system).

Find a symbolic expression for the magnetic field \( \mathbf{B}(0,0,z) \), in Cartesian component form (there are three components), at an arbitrary location on the positive \( z \)-axis.

Note: To receive full credit you must
State the Bio-Savart Law correctly in mathematical form,
work out explicitly the cross product in terms of given parameters and variables,
similarly work out the denominator, and
work out the integral
(5a) Biot-Savart Law:

\[ \mathbf{B} = \frac{\mu_0 I_0}{4\pi} \int \frac{dS \times \hat{r}}{R^2} = \frac{\mu_0 I_0}{4\pi} \int \frac{dS \times \hat{r}}{R^3} \]

\[ dS = Rd\phi \hat{\phi} \]
\[ \phi = -\sin \phi \hat{i} + \cos \phi \hat{j} \]
\[ \mathbf{r} = (0, z) \]
\[ \mathbf{R} = \sqrt{z^2 + R^2} \]
\[ \hat{r} = \frac{1}{\sqrt{z^2 + R^2}} \left[ \hat{z} - \hat{R} \cos \phi \hat{i} - \hat{R} \sin \phi \hat{j} \right] \]

\[ dS \times \hat{r} = Rd\phi \left[ -\sin \phi \hat{i} + \cos \phi \hat{j} \right] \]

\[ \times \frac{1}{\sqrt{z^2 + R^2}} \left[ -\hat{R} \cos \phi \hat{i} - \hat{R} \sin \phi \hat{j} + z \hat{k} \right] \]

\[ = \frac{Rd\phi}{\sqrt{z^2 + R^2}} \left[ R\sin \phi \left( \hat{i} \times \hat{k} \right) - z \sin \phi \left( \hat{i} \times \hat{e} \right) \\ - \hat{R} \cos \phi \left( \hat{j} \times \hat{i} \right) + z \cos \phi \left( \hat{j} \times \hat{k} \right) \right] \]

\[ = \frac{Rd\phi}{\sqrt{z^2 + R^2}} \left[ \hat{z} \cos \phi + \hat{j} \sin \phi + \hat{k} R \left( \sin^2 \phi + \cos^2 \phi \right) \right] \]

Integration limits: \( \phi = -\frac{\pi}{2} \) to \( \phi = +\frac{\pi}{2} \)

\[ \mathbf{B} = \frac{\mu_0 I_0 R}{4\pi (R^2 + z^2)^{3/2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\phi \left[ \hat{z} \cos \phi + \hat{j} \sin \phi + \hat{k} R \right] \]

\[ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\phi \cos \phi = \sin \phi \bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 1 - (-1) = 2 \]

\[ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\phi \sin \phi = -\cos \phi \bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 0 \]

\[ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\phi R = \pi R \]
\[ \mathbf{B} = \frac{M_0 I R}{4\pi (R^2 + z^2)^{3/2}} \left[ \hat{i} 2z + \hat{j} \phi + \hat{k} \frac{I R}{2\pi} \right] \]

Alternatively:

\[ \mathbf{B} = \frac{M_0 I}{4\pi} \int \frac{d\mathbf{s} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^2} \]

\[ \mathbf{r} = (0, 0, z) = z \hat{k} \]

\[ \mathbf{r}' = R \cos \phi \hat{i} + R \sin \phi \hat{j} \]

\[ \mathbf{r} - \mathbf{r}' = -R \cos \phi \hat{i} - R \sin \phi \hat{j} + z \hat{k} \]

\[ |\mathbf{r} - \mathbf{r}'| = (R^2 + z^2)^{1/2} \]

\[ d\mathbf{s} = R d\phi \mathbf{a} = R d\phi (-\sin \phi \hat{i} + \cos \phi \hat{j}) \]

\[ d\mathbf{s} \times (\mathbf{r} - \mathbf{r}') \]

\[ = R d\phi \begin{vmatrix} 1 & 0 & 0 \\ 0 & \sin \phi & \cos \phi \\ 0 & -R \cos \phi & -R \sin \phi \end{vmatrix} \]

\[ = R d\phi \left[ \hat{i} 2 \cos \phi + \hat{j} 2 \sin \phi + \hat{k} R (\sin \phi \hat{i} + \cos \phi \hat{j}) \right] \]

\[ B_x = \frac{M_0 I R^2}{4\pi (R^2 + z^2)^{3/2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin \phi d\phi \cos \phi = \frac{M_0 I R^2}{4\pi (R^2 + z^2)^{3/2}} \left[ \frac{R^2}{2} \sin \phi \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{M_0 I R^2}{2\pi (R^2 + z^2)^{3/2}} \]

\[ B_y = \frac{M_0 I R^2}{4\pi (R^2 + z^2)^{3/2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin \phi d\phi \sin \phi = \frac{M_0 I R^2}{4\pi (R^2 + z^2)^{3/2}} \left[ \frac{R^2}{2} \cos \phi \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 0 \]

\[ B_z = \frac{M_0 I R^2}{4\pi (R^2 + z^2)^{3/2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\pi}{2} d\phi = \frac{M_0 I R^2}{4(R^2 + z^2)^{3/2}} \]
\( F = q \mathbf{v} \times \mathbf{B} \quad \mathbf{v} = V \mathbf{e} \quad \{ V = 3.00 \times 10^6 \text{ m/s} \}
\{ z = z_0 = 0.150 \text{ m} \}

\[
F = q \mathbf{v} \times \frac{\mu_0 I_0 R}{4\pi (R^2 + z_0^2)^{3/2}} \left[ \frac{1}{2} z_0 \times \frac{q}{R \pi R} \right]
\]

\[
= -q V \frac{\mu_0 I_0 R^2}{4(R^2 + z_0^2)^{3/2}} \quad \begin{cases} R = 0.0500 \text{ m} \\ I_0 = 10.0 \text{ A} \\ q = 1.00 \times 10^{-8} \text{ C} \end{cases}
\]

\[
= \frac{(1.00 \times 10^{-8} \text{ C})(3.00 \times 10^6 \text{ m/s})(4\pi (10^7 \text{ A})(1.00)(0.050 \text{ m})^2)}{4(0.0500 \text{ m})^2 + (0.150 \text{ m})^2} \frac{1}{2}
\]

\[
F = -7 \left( 5.96 \times 10^{-8} \text{ N} \right)
\]

\[
F = 59.6 \text{ nN in the -y direction}
\]
4. (10 pts.) Self-Inductance of a Toroid

The toroid shown in the figure below has a triangular cross-section of base \( a \) (located between inner radius \( a \) and outer radius \( 2a \)) and height \( b \) at its highest point. It is wound with a coil of \( N \) turns that carry a current outward at the top.

(a) Use Ampere's Law to find a symbolic expression for the magnetic field \( \vec{B} \) inside the torus as a function of \( r \) (perpendicular distance from the \( z \)-axis). Note: The magnetic field DOES depend on \( r \). If your answer is independent of \( r \) and you use such an answer in the rest of the problem you will receive ZERO credit for parts (c) and (d).

(b) Find a symbolic expression of the height of the triangle, \( h(r) \) as a function of \( r \). Note the height does depend on \( r \), and \( r \) is measured from the center of the torus, NOT the inner edge of the triangle. If you use a height \( h(r) \) that does not depend on \( r \) in the rest of the problem you will receive ZERO credit for parts (c) and (d).

(c) Find a symbolic expression for the flux of the magnetic field through ONE turn of the wires in this toroid.

(d) Find a symbolic expression for the self-inductance \( L \) of this toroid.
From symmetry: \( \mathbf{B} = B(r, z) \hat{\phi} \)

Ampère's Law: \( \oint \mathbf{B} \cdot d\mathbf{s} = N I_{\text{enc}} \)

Take loop \( C \) to be \( \text{ccw} \) at height \( z \) and radius \( r \).

Inside the torus:

\( \mathbf{B} \) is a constant (and in \( \hat{\phi} \) dir.) along \( C \)

and \( d\mathbf{s} \) along \( C \)

\( B \) \( d\mathbf{s} = \phi \) \( r \) \( d\phi \)

\[
\Rightarrow \oint \mathbf{B} \cdot d\mathbf{s} = \int_0^{2\pi} B(r, z) \hat{\phi} \cdot \phi \hat{\phi} \ r d\phi = 2\pi r B(r, z)
\]

For \( C \) inside the loop and \( \text{ccw} \)

\( I_{\text{enc}} = NI_0 \)

\[\Rightarrow 2\pi r B(r, z) = N_0 NI_0 \]

\[\overrightarrow{B}(r, z) = \frac{N_0 NI_0}{2\pi r} \hat{\phi} \]

(b) \( h(r) \)

Note: \( h(r) = 0 \) for \( r < a \)

and \( r \geq 2a \)

for \( a \leq r \leq 2a \), \( h(r=a) = b \) falling linearly

\[ h(r=2a) = 0 \]

--- cont'd
\[ h(r) = A + Br \]
\[ h(a) = A + Ba = b \quad \cdots (1) \]
\[ h(2a) = A + 2Ba = 0 \quad \cdots (2) \]

Taking (1) - (2) \[ B a = b \Rightarrow B = -\frac{b}{a} \]

Substitute back into (1)
\[ \Rightarrow A - \left(\frac{b}{a}\right) a = b \Rightarrow A - b = b \Rightarrow A = 2b \]

\[ h(r) = 2b - \frac{b}{a} r = b \left(2 - \frac{r}{a}\right) \]

4(c)

**da** points into page \( (\phi \ \text{dir}) \)

Break integral into vertical slices
\[ da^2 = h(r) dr \phi^2 \]

\[ \vec{F} = \frac{MonIo}{2\pi} \phi \] also points into page

\[ \Phi_1 = \int B \cdot da^2 = \int_0^{2\pi} \frac{MonIo}{2\pi r} \phi \cdot b \left(2 - \frac{r}{a}\right) dr \]

\[ = 1 \cdot h(r) \]

\[ \Phi_1 = \frac{MonIo b}{2\pi} \int_0^{2\pi} \frac{1}{r} \left(2 - \frac{r}{a}\right) dr \]

\[ \int_0^{2\pi} \frac{1}{r} \left(2 - \frac{r}{a}\right) dr = \int_0^{2\pi} \frac{2}{r} dr - \int_0^{2\pi} \frac{1}{a} dr \]

\[ = 2 \ln \left(\frac{2\pi}{a}\right) - \frac{a}{\pi} = (2\ln 2 - 1) \]

\[ \Phi_1 = \frac{MonIo b}{2\pi} (2\ln 2 - 1) \]
4(a) \[ L = \frac{\Phi_B}{I_0} \quad \overline{\Phi_B} = N \overline{\Phi}_1 \]

\[ \Phi_B = \frac{M_n N^2 I_0 b}{2\pi} (2n^2 - 1) \]

\[ L = \frac{\overline{\Phi_B}}{I_0} = \frac{M_n N^2 b}{2\pi} (2n^2 - 1) \]

\[ L = (0.3615) M_n N^2 b \] \( \leftarrow \) not necessary
5. (10 pts.) Parallel LRC AC circuit.

An \( L = 35 \text{ mH} \) inductor is connected in PARALLEL with a \( R = 20 \text{ \Omega} \) resistor, a \( C = 120 \text{ \mu F} \) capacitor and a \( f = 60 \text{ Hz} \), \( V_{RMS} = 45 \text{ V} \) AC power supply. Since all three devices are in parallel to the supply, they all share the same voltage \( \Delta V \). And the currents through the three devices \( I_L \), \( I_R \), and \( I_C \) add (as phasors/vectors) to give the total current \( I \) put out by the power supply.

(a) Calculate the reactance (in ohms), \( X_L \), \( X_R \), \( X_C \), and the current amplitude \( (I_L)_0 \), \( (I_R)_0 \), and \( (I_C)_0 \) (in amps) of the inductor, resistor and the capacitor, respectively.

(b) Draw a phasor diagram with the voltage \( \Delta V \) along the +x direction, and put in the phasor/vectors representing the currents \( I_L \), \( I_R \), and \( I_C \) through the inductor, resistor and the capacitor, respectively. Also put in the phasor/vector representing \( I \) the total current put out by the AC supply.

(c) For the power supply, we write its voltage as \( V(t) = V_0 \sin \omega t \), and the current it puts out as \( I = I_0 \sin (\omega t - \phi_I) \). Find the RMS current put out by the AC supply, and the phase lag \( \phi_I \) of the current in degrees.

(d) Calculate the average power \( <P> \) generated by the power supply in watts.
5(a) \( f = 60 \text{ Hz} \quad \omega = 2\pi f = 377 \text{ s}^{-1} \)

\[
X_L = \omega L = (377 \text{ s}^{-1})(3.5 \times 10^{-2} \text{ H}) = 13.2 \Omega
\]

\[
X_C = \frac{1}{\omega C} = \frac{1}{(377 \text{ s}^{-1})(1.2 \times 10^{-4} \text{ F})} = 22.1 \Omega
\]

(b) \( \Delta V_o = \sqrt{\Delta V_{\text{RMS}}} = \sqrt{45} V = 6.364 \text{ V} \)

\[
(I_L)_o = \frac{\Delta V_o}{X_L} = \frac{6.364 \text{ V}}{13.2 \Omega} = 0.482 \text{ A}
\]

\[
(I_c)_{\text{RMS}} = \frac{\Delta V_{\text{RMS}}}{X_L} = 3.41 \text{ A}
\]

\[
(I_c)_o = \frac{\Delta V_o}{R} = \frac{6.364 \text{ V}}{20 \Omega} = 0.318 \text{ A}
\]

\[
(I_c)_{\text{RMS}} = \frac{\Delta V_{\text{RMS}}}{R} = \frac{45 \text{ V}}{20 \Omega} = 2.25 \text{ A}
\]

\[
(I_c)_o = \frac{\Delta V_o}{X_C} = \frac{6.364 \text{ V}}{22.1 \Omega} = 0.288 \text{ A}
\]

\[
(I_c)_{\text{RMS}} = \frac{45 \text{ V}}{22.1 \Omega} = 2.04 \text{ A}
\]

\( \Delta V_o / X_L = 4.82 \text{ A} \quad I_L \text{ lags } \Delta V \text{ by } \frac{\pi}{2} \)

Diagram 5-1
\[ I = I_R + I_L + I_C \]
\[ = \frac{\Delta V_0}{R} + \left(\frac{\Delta V_0}{X_C} - \frac{\Delta V_0}{X_R}\right)j \]
\[ = (I_R)_0 + [(I_C)_0 - (I_L)_0]j \]

\[ I_0 = \sqrt{(I_R)_0^2 + [(I_C)_0 - (I_L)_0]^2} \]
\[ = \sqrt{(5.18 \text{A})^2 + (2.88 \text{A} - 4.82 \text{A})^2} \]
\[ = \sqrt{13.876 \text{A}^2} \]
\[ = 3.73 \text{A} \]

\[ I_{RMS} = \frac{1}{\sqrt{2}} I_0 = \frac{1}{\sqrt{2}} (3.73) \text{A} = 2.63 \text{A} \]

\[ \Delta \phi = \tan^{-1} \left( \frac{(I_C)_0 - (I_L)_0}{(I_R)_0} \right) = \tan^{-1} \left( \frac{2.88 - 4.82}{5.18} \right) \]
\[ = \tan^{-1} (-0.610) = -31.4^\circ \]

\( \phi_I \) is the lag of \( I \) from \( \Delta V \)

\[ \phi_I = -\Delta \phi \]
\[ = +31.4^\circ \]

\[ \phi_I \approx 31^\circ \]
- \[ \langle P \rangle = \frac{\Delta V \cdot I}{R_{\text{RMS}}} \]

\[ = (\Delta V_{\text{RMS}})(I_{\text{RMS}}) \cos \phi \]

\[ = (45V)(2.67A) \cos (31.4^\circ) \]

\[ = 101 \text{ W} \]

\[ \boxed{\langle P \rangle = 101 \text{ W}} \]

Alternatively: only the resistor dissipates power

\[ \Rightarrow \langle P \rangle = \frac{1}{2}(I^2)\Delta V_0 = (I_{\text{R}})_{\text{RMS}}(\Delta V_{\text{RMS}}) \]

\[ = (2.25A)(45V) \]

\[ \boxed{\langle P \rangle = 101 \text{ W}} \]

Same answer!
6. (10 pts.) Refraction, Thin Lenses

(a) A light ray is incident from glass $n_1 = 1.48$ on a plane boundary with water $n_2 = 1.33$. What is the smallest angle of incidence $\theta_1$ such that total internal reflection would occur?

(b) An object produces an inverted image, through a converging thin lens, that appears to be one half the height of the object. The distance between the object and the image is 30 cm. What is the focal length (including sign) of the lens?
Snell's Law $n_1 \sin \theta_1 = n_2 \sin \theta_2$

Total internal reflection occurs when $
\sin \theta_2 \geq 1$

Alternately,
The T.I.R. is for

\[ \begin{align*}
\theta_2 &= \frac{\pi}{2} \\
\theta_1 &= \theta_c
\end{align*} \]

\[ n_1 \sin \theta_c = n_2 \left(\frac{n_2}{n_1}\right)^{\frac{1}{2}} \]

\[ \theta_c = \sin^{-1} \left(\frac{n_2}{n_1}\right) \]

\[ \theta_c = \sin^{-1} \left(\frac{1.33}{1.48}\right) \]

\[ = \sin^{-1} (0.8999) \]

\[ = 63.98^\circ \]

\[ \theta_c = 64.0^\circ \]

Is the minimum incident angle for total internal reflection.
6(b) Inverted image, \( \frac{1}{2} \) size

\[
M = -\frac{1}{2} \quad \Rightarrow \quad h' = -\frac{1}{2} h
\]

\[
\frac{h'}{h} = -\frac{q}{p} \quad \Rightarrow \quad -\frac{q}{p} = -\frac{1}{2} \quad \Rightarrow \quad q = \frac{p}{2}
\]

For converging lenses (actually just for lenses)

\( q = \frac{p}{2} \) means the object and image are on opposite sides of the lens.

Distance \( DI = |1| + |q| = p + q = \frac{3}{2} p \)

but \( DI = 0.30 \text{ m} \)

\[
p = \frac{2}{3}(0.30 \text{ m}) = 0.20 \text{ m} = 20 \text{ cm}
\]

\( \frac{q}{p} = \frac{1}{2} p = 0.10 \text{ m} = 10 \text{ cm} \)

\[
\frac{1}{p} + \frac{1}{q} = \frac{1}{f} = \frac{1}{0.20 \text{ m}} + \frac{1}{0.10 \text{ m}} = \frac{3}{0.20 \text{ m}}
\]

\[
f = \frac{0.20 \text{ m}}{3} = 0.0667 \text{ m} = 0.0667 \text{ m} > 0 \text{ for converging lens as expected}
\]

\( f = +0.067 \text{ m} = +6.7 \text{ cm} \)