\( \hat{A}^+ \) ... the only non-zero entry is at \( A_3 = A_3' + 1 \),

i.e., for \( A_3 = \frac{1}{2} \), \( A_3' = -\frac{1}{2} \).

This entry is

\[
\sqrt{\frac{1}{2} \left( \frac{1}{2} + 1 \right) - \left(-\frac{1}{2}\right) \left(-\frac{1}{2} + 1 \right)} = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1.
\]

Hence

\[
\langle A_3 \mid \hat{A}^+ \mid A_3' \rangle = \begin{pmatrix} + & 0 \\ 0 & 1 \end{pmatrix}.
\]

\( \downarrow_{A_3} \)

The Hermitian conjugate yields

\[
\langle A_3 \mid \hat{A}^- \mid A_3' \rangle = \begin{pmatrix} + & 0 \\ 1 & 0 \end{pmatrix}.
\]

\( \downarrow_{A_3} \)

From \( \hat{A}_z = \hat{A}_1 + i \hat{A}_2 \) get \( \hat{A}_2 = \frac{1}{2} (\hat{A}^+ + \hat{A}^-) \),

\( \hat{A}_2 = \frac{1}{2i} (\hat{A}^+ - \hat{A}^-) \).

Hence

\[
\langle A_3 \mid \hat{A}_1 \mid A_3' \rangle = \frac{1}{2} \begin{pmatrix} + & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and}
\]

\( \downarrow_{A_3} \)
\[ \langle \delta_3 | \hat{\alpha}_2 | \delta_3' \rangle = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \]

The \( \delta_3 \)-representation is called the standard representation. Suppose the matrix indices and write

\[ \vec{\beta} = \frac{1}{2} \vec{\delta}, \quad \vec{\delta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

the Pauli matrices.

By their construction, the Pauli matrices are Hermitian.

In the abstract operator form

\[ \hat{\beta}^3 = \frac{1}{2} \hat{\beta}^3 \Rightarrow [\hat{\beta}_1, \hat{\beta}_2] = 2i \hat{\beta}_3 \text{ and cyclic.} \]

In the matrix form

\[ \vec{\beta}^3 = \frac{1}{2} \vec{\delta}^3 \Rightarrow [\delta_1, \delta_2] = 2i \delta_3 \text{ and cyclic.} \]

The matrix commutators.
PROPERTIES OF THE PAULI MATRICES (OPERATORS)

Can be checked from their standard representation, but it is more elegant to work them in the operator form from the basic commutation relations.

\[ \hat{\mathbf{s}}^2 = \frac{3}{4} \mathbf{1} \quad \Rightarrow \quad \hat{s}_1^2 + \hat{s}_2^2 + \hat{s}_3^2 = 3 \mathbf{1} \]

\[ \hat{s}_+^2 = 0 = \hat{s}_-^2 = 0 \quad \Rightarrow \quad \hat{s}_1^2 - \hat{s}_2^2 = i [\hat{s}_1, \hat{s}_2]_+ = 0 \]

(The representation space is 2-dimensional)

\[ \Rightarrow \hat{s}_1^2 = \hat{s}_2^2 \text{ and } [\hat{s}_1, \hat{s}_2]_+ = 0. \]

Of course, all the three axes are on the same footing and hence

\[ \hat{s}_1^2 = \hat{s}_2^2 = \hat{s}_3^2 = \mathbf{1} \text{ and } [\hat{s}_1, \hat{s}_2]_+ = 0 \text{ for } a \neq b. \]

These can be written as a single equation

\[ [\hat{s}_a, \hat{s}_b]_+ = 2 \delta_{ab} \mathbf{1}. \]

Because

\[ [\hat{s}_a, \hat{s}_b] = 2 i \delta_{abc} \hat{s}_c \]
we get
\[ \hat{\sigma}_a \hat{\sigma}_b = \delta_{ab} \hat{1} + i \delta_{abc} \hat{\sigma}_c. \]

All the previous relations are consequences of this single tensor operator equation.

Notice that
\[ \hat{\sigma}_1 \hat{\sigma}_2 = i \hat{\sigma}_3 \] and cyclic, and
\[ \hat{\sigma}_1 \hat{\sigma}_2 \hat{\sigma}_3 = i \hat{1}. \]

**Observables in the Spin Space \( \mathcal{H}(\frac{1}{2}) \).**

The spin space is spanned by the \( |+\rangle \) and \( |-\rangle \) kets.

Observables are represented by \( 2 \times 2 \) Hermitian matrices. The Pauli matrices are Hermitian. So is \( \hat{1} \). The most general Hermitian matrix can be written as their linear combination:

\[
\begin{array}{c|c}
\chi + \chi_3 & \chi_1 - i \chi_2 \\
\hline
\chi_1 + i \chi_2 & \chi - \chi_3
\end{array} = \chi \hat{1} + \chi \cdot \hat{\sigma}. 
\]
$\vec{\sigma}$'s are related to the spin $\vec{s}$ by $\vec{s} = \frac{1}{2} \vec{\sigma}$.

Hence: The only observables in the spin space are linear combinations of the trivial observable $\hat{I}$ and of the angular momentum $\vec{m} \cdot \vec{s}$ about some axis $\vec{m}$:

$$\hat{A} = \alpha \hat{I} + \beta \vec{m} \cdot \vec{s}.$$  

The eigenbasis of $\hat{A}$ coincides with the $|\pm\rangle$ eigenbasis of $\vec{m} \cdot \vec{s}$ (up $\vec{m}$, down $\vec{m}$):

$$\hat{A} |\pm\rangle = (\alpha \pm \frac{1}{2} \beta) |\pm\rangle.$$  

**Rotations in the Spin Space**

$$\hat{U}(\varphi) = e^{-i \varphi \cdot \vec{s}} = e^{-\frac{1}{2} i \varphi \cdot \vec{s}}$$

$$= \cos \left( \frac{1}{2} \varphi \cdot \vec{s} \right) - i \sin \left( \frac{1}{2} \varphi \cdot \vec{s} \right).$$

However,
\[(\vec{\varphi}, \vec{\delta})^2 = \varphi^2 \delta_a \delta_b \frac{\hat{a}_a \hat{a}_b}{\delta_a \delta_b + i \delta_a \delta_c \delta_c} = \varphi^2 \hat{\mathbb{1}}.\]

Therefore

\[
\cos \left( \frac{1}{2} \vec{\varphi} \cdot \vec{\delta} \right) = \cos \frac{1}{2} \varphi \cdot \hat{\mathbb{1}} \quad \text{and} \quad 
\sin \left( \frac{1}{2} \vec{\varphi} \cdot \vec{\delta} \right) = \sin \frac{1}{2} \varphi \cdot (\vec{m} \cdot \vec{\delta}).
\]

This yields

\[
\hat{U}(\vec{\varphi}) = \cos \frac{1}{2} \varphi \cdot \hat{\mathbb{1}} - i \sin \frac{1}{2} \varphi \cdot (\vec{m} \cdot \vec{\delta}).
\]

Here, the unitary operator \(\hat{U}\) is written as a linear combination (with complex coefficients) of the Hermitian operators \(\hat{\mathbb{1}}\) and \(\vec{m} \cdot \vec{\delta}\).

We see that

\[
\hat{U}(2\pi \vec{m}) = -\hat{\mathbb{1}}, \quad \text{but} \quad \hat{U}(4\pi \vec{m}) = \hat{\mathbb{1}}.
\]

The notation by \(2\pi\) changes the phase of the spin state,

\[
\hat{U}(2\pi \vec{m}) |\psi\rangle = -|\psi\rangle,
\]
but it does not change the mean value of any observable \( \hat{A} \) in the spin space:

\[
\langle \psi | \hat{U}^* (2\pi \hat{m}) \hat{A} \hat{U} (2\pi \hat{m}) | \psi \rangle = \langle \psi | (-\hat{\mathbb{I}}) \hat{A} (-\hat{\mathbb{I}}) | \psi \rangle = \langle \psi | \hat{A} | \psi \rangle.
\]

Still, we shall see later that the change of the phase has observable consequences.

**Spinors**

A ket \( \ket{\psi} \in \mathcal{H}(\frac{1}{2}) \) in the spin space which transforms by

\[
\ket{\psi'} = (\hat{\mathbb{I}} \cos \frac{1}{2} \varphi - i (\vec{\hat{m}} \cdot \vec{\hat{S}}) \sin \frac{1}{2} \varphi) \ket{\psi}
\]

under rotations \( R(\varphi \hat{m}) \) is called a spinor.

Spinors can be represented by their components \( \psi^A := \langle A | \psi \rangle \) in an arbitrary orthonormal basis \( \ket{A} \), \( A = 1, 2 \), in the spin space \( \mathcal{H}(\frac{1}{2}) \).
(Remember that any such basis is the eigenbasis \( | \pm \rangle \) of some spin operator \( \vec{m} \).)

The norm of \( | \psi \rangle \) can be written in the form

\[
\langle \psi | \psi \rangle = \sum_{A=1}^{2} \left( \frac{\langle \psi | A \rangle \langle A | \psi \rangle}{(\psi^A)^* \psi^A} \right).
\]

To take advantage of the summation convention, write

\[
\psi^*_A := (\psi^A)^* = \langle \psi | A \rangle.
\]

Then

\[
\langle \psi | \psi \rangle = \psi^*_A \psi^A.
\]

Find how \( \psi^A \) and \( \psi^*_A \) transform under notations:

\[
| \psi' \rangle = \hat{U} | \psi \rangle \Rightarrow
\]

\[
\langle A | \psi' \rangle = \sum_{B=1}^{2} \left( \frac{\langle A | \hat{U} | B \rangle \langle B | \psi \rangle}{\psi^*_A U^A_B \psi^B} \right),
\]
\[ \psi^* A = U^A_B \psi^B \]

The operator \( \hat{O}^* \) adjoint to \( \hat{O} \) is represented by the matrix \( U^B_A^* \) which is Hermitian conjugate of the matrix \( U^A_B \):

\[ (U^A_B)^* = \langle A | \hat{O} | B \rangle^* = \langle B | \hat{O}^* | A \rangle = : U^B_A^* : \]

Because \( \hat{O} \) is unitary,

\[ U^B_A^* = U^{-1} B_A^* \]

Work now the transformation property of \( \psi^* A \):

\[ \langle \psi' | = \langle \psi | \hat{O}^* = \right\]

\[ \langle \psi' | A \rangle = \sum_{B=1}^{2} \langle \psi | B \rangle \langle B | \hat{O}^* | A \rangle , \text{ i.e.,} \]

\[ \psi^*' A = \psi^*_B U^B_A = U^{-1} B_A \]

\[ \psi^*'_A = U^{-1} B_A \psi^* B \], or \[ \psi^*_B = U_B^A \psi^*_A \]
Conjugate spinors $\psi^* a$ thus transform under notations by the inverse matrix. We see that $\psi^A$ is a contravariant spinor, $\psi^* a$ is a covariant spinor.

There is still another way of converting spinors into cospinors. It is based on the fact that the rotation matrices are not arbitrary unitary matrices, but are unimodular:

$$|U^A_B| = \det(U^A_B) = 1.$$  

Notice first that the determinant of the matrix $F^A_B = \langle A | \hat{F} | B \rangle$ which represents an arbitrary operator $\hat{F}$ does not depend on the choice of the orthonormal basis. We know that if we change the basis, $|A'\rangle = \hat{W} | A \rangle$, this happens by a unitary operator $\hat{W}$, and that

$$F'^{A'}_{B'} = W^A_{A'} F^A_B W^{-1}_{B'} B.$$
Taking the determinants

\[ |F'_{A'B'}| = |W_A'| |F_B| |W^{-1}_{B'}| = |F_B|. \]

These determinants are inverse numbers.

We are thus entitled to evaluate \( U^A_B (\eta \vec{m}) \) in any basis. Let us do it in the basis of the \( \vec{m} \), \( \vec{\sigma} \) operator. Then

\[
U^A_B (\eta \vec{m}) = \begin{pmatrix}
\frac{1}{2} \cos \frac{1}{2} \eta - i \sigma_3 \sin \frac{1}{2} \eta \\
\cos \frac{1}{2} \eta - i \sin \frac{1}{2} \eta \\
\cos \frac{1}{2} \eta + i \sin \frac{1}{2} \eta \\
0
\end{pmatrix}
= \begin{pmatrix}
e^{-\frac{1}{2} i \eta} & 0 \\
0 & e^{\frac{1}{2} i \eta}
\end{pmatrix}
\]

and

\[ |U^A_B| = e^{-\frac{1}{2} i \eta} e^{\frac{1}{2} i \eta} = 1. \]

The unimodular unitary matrices form a group \( SU(2) \) (special 2x2 unitary matrices).