\[ \psi'^A(x) = \langle \hat{x}, A | \hat{U} | \psi \rangle = U_B^A \psi^B(R(-\varphi^2) \hat{x}). \]

**Magnetic Moment of Spin \( \frac{1}{2} \) Particles**

**Orbital Magnetic Moment**

\[ I = \frac{e}{\eta^2} = \frac{e}{2\pi^2} \frac{1}{\nu} = \frac{eU}{2\pi^2} \]

**Total Current**

\[ \mu = \frac{I}{c} \Sigma = \frac{1}{c} \frac{eU}{2\pi^2} \nu \pi^2 \]

**Magnetic Moment of the Electric Loop**

\[ \vec{\mu} = \frac{e}{2mc} \vec{L} \]

relates the magnetic moment to the angular momentum of a particle in orbital motion.

For \( e > 0 \), \( \vec{\mu} \parallel \vec{L} \),

for \( e < 0 \), \( \vec{\mu} \perp \vec{L} \).
In quantum theory, \( \hat{\mu} \) is quantized and hence \( \hat{\mu} \) is also quantized:

\[
\hat{\mu} = \frac{e}{2mc} \hat{\mathbf{e}} = \frac{1e\hbar}{2mc} (\text{sign} e) \frac{\hat{\mathbf{e}}}{\hbar}
\]

For the eigenvalues

\[
\mu_3 = \frac{|e|\hbar}{2mc} (\text{sign} e) m
\]

\[
\downarrow \quad \text{mass}
\]

\[
\text{MAGNETON of a particle}
\]

\[
\text{magnetic quantum number} \quad \ell = 0, \pm 1, \ldots, \pm \ell
\]

For an electron, \( \mu_B = \frac{1e\hbar}{2mc} \) is called Bohr's magneton.

**INTRINSIC MAGNETIC MOMENT OF AN ELECTRON**

Experiments show that electrons have an intrinsic magnetic moment which is again proportional to the intrinsic angular momentum (the spin) of the electron:

\[
\hat{\mu} = -2\mu_B \hat{\mathbf{s}} = -\mu_B \frac{\hat{\mathbf{s}}}{\hbar}
\]

\[
\downarrow 9 \quad \text{GYROMAGNETIC RATIO}
\]
For an electron, $e < 0$, the intrinsic magnetic moment is antiparallel to the spin. Dirac's relativistic Theory of the electron explains the factor 2.

**INTRINSIC MAGNETIC MOMENT OF NUCLEONS**

For nucleons, the magnetic moment is customarily measured in nuclear magnetons

\[
\mu_N = \frac{1e\hbar}{2mp_c} \quad (mp \text{ mass of the proton}).
\]

For a proton

\[
\mu = 5.59 \frac{1e\hbar}{2mp_c} \hat{\text{A}}_{\parallel g \text{ (proton)}}
\]

For a neutron

\[
\mu = -3.83 \frac{1e\hbar}{2mp_c} \hat{\text{A}}_{\parallel g \text{ (neutron)}}
\]
Because $\mu_B \approx 2000 \mu_N$, in the g units electrons have a much \textbf{LARGER} magnetic moment than the nucleons.

Higher spin particles may also have an \textit{intrinsic} magnetic moment.

\textbf{DIRECT EXPERIMENTS MEASURING THE MAGNETIC MOMENT AND THE SPIN OF ELECTRONS}

\textbf{Magnetic moment: STERN - GERLACH (1922)}

H atom in its ground state (actually, Ag)

\begin{itemize}
  \item Inhomogeneous magnetic field, $B \parallel z$ growing in the $z$-direction
\end{itemize}

\begin{equation}
U = -\vec{\mu} \cdot \vec{B} = -\mu_B B_z \text{ energy of a magnetic dipole}
\end{equation}

\begin{equation}
\vec{F} = -\nabla U \text{ The force on the dipole}
\end{equation}
\[ F_z = \mu_z \frac{\partial B_z}{\partial z} = -2 \mu_B \frac{\partial B_z}{\partial z} N_z \]

\( \rightarrow \) magnetic spin number, \( \pm \frac{1}{2} \)

Particles with spin up go down, particles with spin down go up.

Spin: EINSTEIN - DE HAAS (1925)

Homogeneous magnetic field

\[ \vec{\mu} \text{'s of the electron aligned along } \vec{B}, \text{ giving rise to the total magnetic moment } \vec{M}, \text{ spins antialigned, leading to the spin angular momentum } \vec{S} \]

\[ \vec{B} \]

Magnetic field suddenly switches direction, leading to the realignment of magnetic moments and the spins.

\[ \vec{B} \]

\[ \Delta S = 2S \] by conservation of the total angular momentum leads to the sample turning clockwise.
THE PAULI EQUATION

The Schrödinger equation for a non-relativistic particle with spin $\hat{S}$ which is moving in an external electromagnetic field.

(Written by Pauli originally for an electron.)

Recall that $\vec{\mu} = \mu \hat{S}$.

$- \vec{\mu} \cdot \vec{B}(x)$ ... potential energy of a magnetic dipole $\vec{\mu}$ in the magnetic field $\vec{B}$

$= - \mu \hat{S} \cdot \vec{B}$ for a spinning particle.

This leads to the Hamilton operator

$\hat{H} = \left( \frac{1}{2m} \hat{P}^2 - \frac{e}{c} \hat{A} \cdot \vec{A} \right) \hat{1} - \mu \hat{S} \cdot \vec{B}$.

unit operator in the spin space
In particular, for electrons \( e = -\frac{|e|}{e} \) and elementary charge

\[
\hat{H} = \frac{1}{2m} \left( |\hat{p}|^2 + \frac{|e|}{c} |\hat{A}|^2 - |e| |\hat{\phi}|^2 \right) \mathbb{1} + \mu_B \overrightarrow{B} \cdot \vec{\sigma}.
\]

\( \vec{\sigma} \rightarrow \) Bohr's magneton

The \textbf{Pauli equation}:

\[
\text{i} \hbar \frac{d}{dt} \left| \psi(t) \right\rangle = \hat{H} \left| \psi(t) \right\rangle.
\]

Here, \( \left| \psi \right\rangle \in \mathcal{H}(\frac{1}{2}) \) space spanned by \( |x, s_3\rangle \).

\( \mathcal{H}(\frac{1}{2}) \) mixes the components of the spinor field belonging to different values of \( s_3 \).

**Spin Precession**

A spin-\( \frac{1}{2} \) particle of charge \( e \) moves in a homogeneous magnetic field \( \overrightarrow{B} \) directed along \( \mathbb{1} \).

The vector potential \( \vec{A} \) and scalar potential \( \overrightarrow{\Phi} \).

The Hamiltonian \( \hat{H} \):

\[
\hat{H} = \left\{ \frac{1}{2m} \left( |\hat{p}|^2 - \frac{e}{c} |\hat{A}|^2 \right) \mathbb{1} - \mu B \overrightarrow{B} \cdot \vec{\sigma} \right\} + \mu B \overrightarrow{B} \cdot \vec{\sigma},
\]

The spin depends only on \( \vec{r} \) and \( \hat{p} \).

The scalar potential \( \overrightarrow{\Phi} \).
The Pauli equation for the state function
\[ \langle A, \vec{x} | \psi \rangle = \psi^A(\vec{x}) \] separates:

\[ \psi^A(\vec{x}, t) = \frac{u(\vec{x}, t)}{\phi^A(t)} \]

**ORBITAL PART**  **SPIN PART**

The orbital part satisfies the Schrödinger equation
\[ i\hbar \frac{\partial u(\vec{x}, t)}{\partial t} = \hat{H}_{\text{orb}} u(\vec{x}, t) \]
and the spin part satisfies the equation
\[ i\hbar \frac{\partial \phi^A(t)}{\partial t} = -\mu B \left( \vec{m} \cdot \vec{\alpha} \right)^A_B \phi^B(t). \]

Abstractly,
\[ \hat{H}_{\text{spin}} \]

\[ i\hbar \frac{\partial}{\partial t} | \phi(t) \rangle = \hat{H}_{\text{spin}} | \phi(t) \rangle. \]

The evolution operator \( \hat{U}(t, 0) \) of the spin state is
\[ \hat{U}(t, 0) = e^{-\frac{i}{\hbar} \hat{H}_{\text{spin}} t} = e^{\frac{i}{2} \frac{\mu B}{\hbar} (\vec{m} \cdot \vec{\alpha}) t} \]

It is identical to the rotation operator \( \hat{R}(\varphi, \vec{\alpha}) \)

with \( \varphi = -\frac{\mu B}{\hbar} t \)

\( \omega \) ... a frequency
It transforms the initial state $|\phi\rangle := |\phi(0)\rangle$

into the final state

$$|\phi(t)\rangle = \hat{U} \hat{R} \left( \varphi, \vec{n} \right) |\phi\rangle.$$ 

We know that $|\phi\rangle$ must be an eigenvector of the spin operator $\hat{\vec{s}} \cdot \vec{m}$ in some direction $\vec{m}$, and similarly $|\phi(t)\rangle$ must be an eigenvector of the spin operator $\hat{\vec{s}} \cdot \vec{m}(t)$ in some direction $\vec{m}(t)$. It is easy to see that

$$m_a(t) = R^a_b \left( \varphi, \vec{n} \right) m_b.$$ 

We can interpret this equation as saying that the spin rotates about the direction $\vec{m}$ of the magnetic field with the angular velocity $\omega = -\mu B / \hbar$.

The last equation can also be written as

$$\vec{m}(t) = \vec{m} \cos \varphi + (\vec{m} \cdot \vec{n}) \vec{n} (1 - \cos \varphi) + (\vec{n} \times \vec{m}) \sin \varphi.$$
After the time $\Delta t = \frac{2\pi}{\omega}$, the spin returns to its original direction, but the spin state vector changes sign,

$$|\phi(\frac{2\pi}{\omega})\rangle = -|\phi(t)\rangle.$$  

We want to know if such a change of sign is observable.

There is an alternative way of viewing the same problem. We do not separate $\vec{E}$ and $A$, but solve the unseparated equation in the $\vec{E}$, $\vec{A}$ representation:

Write

$$\hat{H} = \hat{H}_{\text{orb}} + \hat{H}_{\text{spin}}, \quad \text{with} \quad \hat{H}_{\text{spin}} = -\mu \vec{B} \cdot \vec{S}.$$  

Because $\hat{H}_{\text{orb}}$ and $\hat{H}_{\text{spin}}$ commute, we can write

$$\psi^A (\vec{x}, t) = \langle A | e^{-\frac{i}{\hbar} \hat{H}_{\text{spin}} t} | B \rangle e^{\frac{i}{\hbar} \hat{H}_{\text{orb}} t} \psi^B (\vec{x}, 0)$$

This is the state function as it would evolve in the absence of the Pauli term $\hat{H}_{\text{spin}}$.  

$$\psi^A (\vec{x}, t)$$

is called $\psi^A (\vec{x}, t)$.
We have

\[
U_B^A(t) = e^{ \frac{i}{2} \omega t } e^{ - \frac{i}{2} \omega t }.
\]

Hence

\[
\psi^+(\vec{x}, t) = e^{ \frac{i}{2} \omega t } \psi^0_0(\vec{x}, t),
\]

\[
\psi^-(\vec{x}, t) = e^{ \frac{i}{2} \omega t } \psi^0_0(\vec{x}, t).
\]

This is, of course, a superposition of solutions of the separated equation.

**Neutron Interferometry Experiment**

For Detecting the Sign Change under 2π Rotations

Let us return now to the question if the sign change of the state function after \( \Delta t = \frac{2\pi}{\omega} \) is observable.