1) Introduction

Ordinary differential equations of any order can be rewritten as a system of first order equations.

For example: \( \frac{dy}{dx}(x) + p(x) \frac{dz}{dx}(x) + q(x) y(x) = r(x) \) is equivalent to the system:

\[
\begin{align*}
\frac{dy}{dx}(x) &= \frac{dz}{dx}(x), \\
\frac{dz}{dx}(x) &= f(x, y, z, \ldots, y_n) \quad \text{with} \quad i = 0, 1, 2, \ldots, n-1
\end{align*}
\]

As a consequence, we consider problems that can be written as \( \frac{dy}{dx}(x) = f(x, y_1, y_2, \ldots, y_n) \) with \( i = 0, 1, 2, \ldots, n-1 \).

For a problem to be completely defined, we need boundary conditions: values of \( y_i \) to be reached at specific values of \( x \).

This results in two types of problems:

- **Initial value condition problems**: all the constraints on \( y_i \) are specified for a single value of \( x = x_0 \).
- **Boundary value problems**: The constraints are specified in two or more values of \( x \).

We concentrate first on initial value problems.

2) Euler's method

\( y_i(x+(k+1)h) = y_i(x+kh) + h \frac{dy}{dx}(x+kh) \quad \text{where} \quad i = 1, 2, \ldots, n \)

Example:

\[
\frac{dy}{dx} = g, \quad \frac{dz}{dx} = g \quad \text{with initial conditions} \quad y(0) = 0, \quad z(0) = 0
\]

With \( g = 100 \times x^2 \) & using time steps \( h = 0.1 \).

<table>
<thead>
<tr>
<th>( k )</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n_k )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>( y_k )</td>
<td>0</td>
<td>0.1</td>
<td>0.2</td>
<td>0.3</td>
<td>0.6</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Initial conditions

It is clear that adding a higher order term \( \frac{d^2 y}{dx^2}(x) \) at each step improves the result (making it exact in the case of a uniformly accelerated motion).

3) Modified Euler's method (predictor-corrector)

The development can indeed generally extended to higher order:

\( y_j(t+h) = y_j(t) + h \frac{dy}{dx}(t) + \frac{h^2}{2} \frac{d^2 y}{dx^2}(t) + \cdots + \frac{h^n}{n!} \frac{d^n y}{dx^n}(t) + \cdots \)

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Using a development involving \( \frac{d^2 y}{dx^2}(t) \) results in an order \( p+2 \) of the error.

The use of such development suffers from the fact it uses the derivatives only at the beginning of the interval \( [t, t+h] \).

It is possible to get an higher order with less work: Consider for example the development around \( t + \frac{h}{2} \):

\[
\begin{align*}
\{ y_j(t+h) - y_j(t) &= \frac{h}{2} \frac{dy}{dx}(t + \frac{h}{2}) + \frac{h^2}{8} \frac{d^2 y}{dx^2}(t + \frac{h}{2}) + \cdots \\
\{ y_j(t) = y_j(t) - \frac{h}{2} \frac{dy}{dx}(t + \frac{h}{2}) + \frac{h^2}{8} \frac{d^2 y}{dx^2}(t + \frac{h}{2}) + \cdots
\end{align*}
\]

Subtracting the two \( y_j(t+h) - y_j(t) = h \frac{dy}{dx}(t + \frac{h}{2}) + O(h) \)
Then, we can use 
\[
\frac{dy}{dt} (t + \frac{1}{2}) = \frac{1}{2} \left( f(t, y(t)) + f(t + h, y(t + h)) \right)
\] and then:

\[
y(t + h) = y(t) + \frac{h}{2} \left[ f(t, y(t)) + f(t + h, y(t + h)) \right] + O(h^3)
\]

\(y(t + h)\) appears on both sides of the equation.

- Sometimes it is possible to solve for \(y(t + h)\)
- Otherwise, one may use Euler's method in the right hand:

\[
y(t + h) = y(t) + \frac{h}{2} \left( f(t, y(t)) + f(t + h, y(t + h)) \right) + O(h^3)
\]

Which is the modified Euler or predictor-corrector method.

This is a special case of Runge-Kutta methods, with two stages only.

The commonly used 4-stage Runge-Kutta method proceeds as follows:

\[
k_1 = f(t, y(t)); k_2 = f(t + \frac{h}{2}, y(t) + \frac{h}{2} k_1); k_3 = f(t + \frac{h}{2}, y(t) + \frac{h}{2} k_2); k_4 = f(t + h, y(t) + h k_3)
\]

\[
y(t + h) = y(t) + \frac{h}{6} (k_1 + 2k_2 + 2k_3 + k_4) + O(h^5)
\]

When \(f\) depends on \(t\) only, it is the same as Simpson's rule.

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**Simple pendulum**

As an example of differential equation integration, we are going to consider the simple pendulum.

We can obtain the equation of motion in two equivalent ways:

1. **Fundamental relation of dynamics along direction \(\hat{\theta}\):**

   \[ m \ddot{\theta} = \frac{d}{dt} (m \ell \dot{\theta}) = -mg \sin \theta \]

   So, \[ \frac{d^2 \theta}{dt^2} = - \frac{g}{\ell} \sin \theta \]

2. **Torque & moment of inertia:**

   \[ \tau = \ell \ddot{\theta} \]

   with \( \tau = \frac{1}{2} m \ell^2 \) the pendulum moment of inertia

   and \( \ddot{\theta} \) the angular acceleration pointing out of the diagram, along \( \hat{\theta} \) for counterclockwise angular accelerations.

3. The \(z\)-component of this equation then is \[ m \ell \dddot{\theta} = - \ell m g \sin \theta \]

   and again \[ \frac{d^2 \theta}{dt^2} = - \frac{g}{\ell} \sin \theta \]

4. In the small angle limit \( \theta \ll 1 \), this becomes \[ \frac{d^2 \theta}{dt^2} = - \frac{g}{\ell} \theta \]

5. The 2nd order differential equation can be recast in a system of two 1st order equations:

   For this system, an initial value problem can be specified by \( \theta_0(t=0) \) & \( \dot{\theta}(t=0) \): \[
   \begin{align*}
   \frac{d\theta}{dt} &= \theta_0 \\
   \frac{d\theta_0}{dt} &= -\omega^2 \sin \theta
   \end{align*}
   \]