Fibonacci Numbers and Chebyshev Polynomials

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Abstract

The relation between the Fibonacci Sequence and the Golden Ratio is quite intriguing, given that the sequence of integer is somehow closely related to the irrational number. More surprisingly, Chebyshev Polynomial of the second kind is also deeply connected to the Fibonacci Sequence. In this article, we investigate the relation between those seemingly unrelated topics.

1 Fibonacci Sequence and Golden Ratio

1.1 Recurrence Relation

The Fibonacci numbers: 0, 1, 1, 2, 3, 5, · · · , are the sequence of numbers defined by the linear recurrence equation:

\[ f_n = f_{n-1} + f_{n-2} \]  \hspace{1cm} (1)

To find \( f_n \), we first rewrite Eq. 1 in the matrix form:

\[
\begin{bmatrix}
    f_n \\
    f_{n-1}
\end{bmatrix}
= M \begin{bmatrix}
    f_{n-1} \\
    f_{n-2}
\end{bmatrix} = \cdots = M^{n-1} \begin{bmatrix}
    f_1 \\
    f_0
\end{bmatrix}, \text{ where } M = \begin{bmatrix}
    1 & 1 \\
    1 & 0
\end{bmatrix}.
\]

Noting that \( f_0 = 0 \) and \( f_1 = 1 \), we obtain \( f_n = (M^n)_{1,2} = (M^{n-1})_{1,1} \).

The characteristic equation of \( M \) is given by

\[ \lambda^2 - \lambda - 1 = 0, \]  \hspace{1cm} (2)

and by solving Eq. 2, you obtain the eigenvalues

\[ \lambda_{\pm} = \frac{1 \pm \sqrt{5}}{2}, \]  \hspace{1cm} (3)
and the closed form of $f_n$ is therefore given by

$$f_n = \frac{\lambda^+_n - \lambda^-_n}{\sqrt{5}} \tag{4}$$

### 1.2 The “Golden Ratio”

$\lambda_+ = (1 + \sqrt{5})/2 = 1.6180\cdots$ is widely acknowledged as the *Golden Ratio*, often denoted as $\varphi$. Since $\varphi$ is the roots of Eq. 2, it can be expressed in a nested radical representation:

$$\varphi = \sqrt{1 + \varphi} = \cdots = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}} \tag{5}$$

and can also be expressed in the form of continued fraction:

$$\varphi = 1 + \frac{1}{\varphi} = \cdots = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}} \tag{6}$$

The golden ratio appears frequently in nature [1] (e.g. in the structure of crystals), and the special case of logarithmic spirals

$$r(\theta) = \exp\left(\frac{2\ln(\varphi)}{\pi}\theta\right), \tag{7}$$

is often called the *Golden Spiral*, which can be observed in the arms of the spiral galaxies as illustrated in Fig. 1, a shape nautilus shell and so forth.

### 2 Chebyshev Polynomial

#### 2.1 Chebyshev Polynomial of the first kind

The Chebyshev polynomials of the first kind\(^1\), $T_n(x)$, defined as

$$\frac{1 - tx}{1 - 2tx + t^2} = \sum_{n=0}^{\infty} T_n(x)t^n, \tag{8}$$

\(^1\)For more details, see Ref.[2]
The Golden Spiral
\[ r(\theta) = \exp\left(\frac{2\ln(\phi)\theta}{\pi}\right) \]

Figure 1: The left panel is the image of the Whirlpool Galaxy (M51), as a spectacular example of the appearance of the golden ratio in nature. The right panel is the plot of Eq. 7 in polar coordinates for \( \theta \in [0, 4\pi] \).

are a set of orthogonal polynomials:
\[
\int_{-1}^{1} d\frac{T_m(x)T_n(x)}{\sqrt{1-x^2}} = \begin{cases} \frac{1}{2}\pi\delta_{m,n} & \text{for } m \neq 0, n \neq 0 \\ \pi & \text{for } m = n = 0 \end{cases}
\] (9)

The first few polynomials are shown in Fig. 2 for \( n = 1, \cdots, 5 \) and \( x \in [-1,1] \). They satisfy the recurrence relations:
\[
T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).
\] (10)

They appear in many branches of mathematics, but most commonly known to be connected with trigonometric multiple-angle formulas
\[
\cos(n\theta) = T_n(\cos(\theta)).
\] (11)

2.2 Chebyshev Polynomial of the second kind

The Chebyshev polynomials of the second kind\(^2\), \( U_n(x) \), defined as
\[
\frac{1}{1-2tx+t^2} = \sum_{n=0}^{\infty} U_n(x)t^n,
\] (12)

\(^2\)For more details, see Ref.[2]
The first few Chebyshev polynomials of the first kind $T_n(x)$ for $n = 1, \ldots, 5$ and $x \in [-1, 1]$.

are a set of orthogonal polynomials:

$$\int_{-1}^{1} dx U_m(x)U_n(x)\sqrt{1 - x^2} = \frac{\pi}{2} \delta_{m,n}$$

They satisfy the same recurrence relations with $T_n(x)$:

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x).$$

with a slightly different initial conditions.

3 The Connection between the Fibonacci Sequence and the Chebyshev polynomials of the second kind

The Fibonacci numbers can be expressed in terms of the Chebyshev polynomial of the second kind by

$$f_{n+1} = i^{-n}U_n(i/2).$$
Figure 3: The number of square-domino fillings of length four: with no weight, this corresponds to $f_5 = 4C_0 + 3C_1 + 2C_2 = 5$, whilst with the weight of $2x$ and $-1$ for square and domino respectively, this corresponds to $U_4(x) = 4C_0(2x)^4 + 3C_1(-1)(2x)^2 + 2C_2(-1)^2 = 16x^4 - 12x^2 + 1$.

This rather bizarre relation has an elegant proof led by combinatorial models[3]. In the combinatorial model, the Fibonacci number $f_{n+1}$ counts the ways to fill a $1 \times n$ stripe using $1 \times 1$ square and $1 \times 2$ dominos. As it turns out, Chebyshev polynomials counts the same objects as the Fibonacci numbers, with an additional weight to each square and domino. More specifically, each square tile and domino are assigned a weight of $2x$ and $-1$ respectfully. Fig. 3 illustrates the counting of tilling of the length four stripe, which, with no weighting, corresponds to $f_5$ and which, with the weighting mentioned above, corresponds to $U_4(x)$. If we take $x = i/2$, we see that every tiling of length $n$ with $k$ dominos has weight $(-1)^k i^{n-2k} = i^n$, which turns out to be independent of $k$. Since $f_n$ is the tiling of of length $n$, $U_n(i/2) = i^n f_{n+1}$, which provides the proof of Eq. 15.

References

