Fibonacci Numbers and Chebyshev Polynomials

Takahiro YAMAMOTO

Department of Physics and Astronomy, University of Utah, USA

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Abstract

The relation between the Fibonacci Sequence and the Golden Ratio is seemingly unrelated topics.

1 Fibonacci Sequence and Golden Ratio

1.1 Recurrence Relation

The Fibonacci numbers: 0, 1, 1, 2, 3, 5, · · · , are the sequence of numbers defined by the linear recurrence equation:

\[ f_n = f_{n-1} + f_{n-2}. \]  

(1)

To find \( f_n \), we first rewrite Eq. 1 in the matrix form:

\[
\begin{bmatrix}
  f_n \\
  f_{n-1}
\end{bmatrix}
= M
\begin{bmatrix}
  f_{n-1} \\
  f_{n-2}
\end{bmatrix}
= \cdots = M^{n-1}
\begin{bmatrix}
  f_1 \\
  f_0
\end{bmatrix}, \text{ where } M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.
\]

Noting that \( f_0 = 0 \) and \( f_1 = 1 \), we obtain \( f_n = (M^n)_{1,2} = (M^{n-1})_{1,1} \).

The characteristic equation of \( M \) is given by

\[ \lambda^2 - \lambda - 1 = 0, \]  

(2)

and by solving Eq. 2, you obtain the eigenvalues

\[ \lambda_\pm = \frac{1 \pm \sqrt{5}}{2}, \]  

(3)

and the closed form of \( f_n \) is therefore given by

\[ f_n = \frac{\lambda_+^n - \lambda_-^n}{\sqrt{5}} \]  

(4)
The “Golden Ratio”

$\lambda_+ = (1 + \sqrt{5})/2 = 1.6180\cdots$ is widely acknowledged as the Golden Ratio, often denoted as $\varphi$. Since $\varphi$ is the roots of Eq. 2, it can be expressed in a nested radical representation:

$$\varphi = \sqrt{1 + \varphi} = \cdots$$

(5)

and can also be expressed in the form of continued fraction:

$$\varphi = 1 + \frac{1}{\varphi} = \cdots$$

(6)

The golden ratio appears frequently in nature [1] (e.g. in the structure of crystals), and the special case of logarithmic spirals

$$r(\theta) = \exp\left(\frac{2\ln(\varphi)}{\pi} \theta\right),$$

(7)

is often called the Golden Spiral, which can be observed in the arms of the spiral galaxies as illustrated in Fig. 1, a shape nautilus shell and so forth.

2 Chebyshev Polynomial

2.1 Chebyshev Polynomial of the first kind

The Chebyshev polynomials of the first kind $\cdots$

The first few polynomials are shown in Fig. 2 for $n = 1, \cdots, 5$ and $x \in$
Chebyshev Polynomial of the First Kind

Figure 2: The first few Chebyshev polynomials of the first kind $T_n(x)$ for $n = 1, \cdots, 5$ and $x \in [-1,1]$.

They appear in many branches of mathematics, but most commonly known to be connected with trigonometric multiple-angle formulas

$$\cos(n\theta) = T_n(\cos(\theta)).$$

(8)

2.2 Chebyshev Polynomial of the second kind

The Chebyshev polynomials of the second kind\(^1\), $U_n(x)$, defined as

$$\frac{1}{1 - 2tx + t^2} = \sum_{n=0}^{\infty} U_n(x)t^n,$$

(9)

are a set of orthogonal polynomials:

$$\int_{-1}^{1} dx U_m(x)U_n(x)\sqrt{1-x^2} = \frac{\pi}{2}\delta_{m,n}$$

(10)

\(^1\)For more details, see Ref.[2]
They satisfy the same recurrence relations with $T_n(x)$:

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x).$$  \hspace{1cm} (11)

with a slightly different initial conditions.

3 The Connection between the Fibonacci Sequence and the Chebyshev polynomials of the second kind

The Fibonacci numbers can be expressed in terms of the Chebyshev polynomial of the second kind by

$$f_{n+1} = i^{-n}U_n(i/2).$$ \hspace{1cm} (12)

This rather bizarre relation has an elegant proof led by combinatorial models\[3\].

References

