A solid, uniform cylinder of mass $m$ and radius $R$ is fitted with a frictionless axle along the cylinder's long axis. A horizontal spring (spring constant $k$) is attached to this axle. Under the influence of the spring, the cylinder rolls back and forth without slipping on a horizontal surface. What is the frequency of this motion?

Work this problem in two different ways: (i) using Newton's laws of motion in translational and rotational form, and (ii) using the conservation laws. Doing the same problem two ways lets you see how the two methods are equivalent (in that they give you the same end results), but have different advantages and disadvantages in their implementation.

(i) Using the sum of the torques:

$$
\Sigma \tau = I' \alpha = \bar{R} \times \bar{F}
$$

Here, the spring acts on the center of mass of the cylinder, but the torque is applied at the bottom of the cylinder. Thus, we need to use the parallel-axis theorem:

$$
I' = I_{cm} + mR^2 = \frac{1}{2} mR^2 + mR^2 = \frac{3}{2} mR^2
$$

$$
\tau = I' \alpha = RF
$$

$$
\Rightarrow (\frac{3}{2} mR^2) \frac{\dot{x}^2}{R} = \frac{k}{R} (-kx) \Rightarrow \dot{x} = -\frac{2k}{3m} x \Rightarrow \omega = \sqrt{\frac{2k}{3m}}
$$

(ii) Using conservation of energy:

$$
\frac{1}{2} I \omega^2 + \frac{1}{2} m \dot{v}^2 = -\frac{1}{2} kx^2
$$

$$
\Rightarrow (\frac{1}{2} mR^2) \frac{v^2}{R^2} + mv^2 = -kx^2
$$

Take derivative:

$$
\frac{d}{dt} \left( \frac{1}{2} m \dot{x}^2 + m \dot{x}^2 \right) = -\frac{d}{dt} kx^2
$$

where $v = \dot{x}$ and $a = \ddot{x}$

$$
\Rightarrow \dot{x} \ddot{x} = -2kx \dot{x}
$$

$$
\Rightarrow \dot{x} = -\frac{2k}{3m} x \Rightarrow \omega = \sqrt{\frac{2k}{3m}} \checkmark \text{Same answer}
$$