Problem 1.

a. Electric field and force. An electric dipole consists of two charges, \( q = \pm 6.7 \times 10^{-16} \text{ C} \), separated by 12 nm and centered at the origin. If the dipole moment is aligned with the y-axis, as shown, Find the electric field \( \vec{E} \) at a point \( P \) that lies on the x-axis at a distance of 20 nm away from the origin, and give the force \( \vec{F} \) on an electron at that point.

\[
\gamma = (20 \hat{e}_x \pm 6 \hat{e}_y) \text{(nm)} \quad \gamma^2 = 436 \text{(nm)}^2
\]

\[
\vec{E}_z = \frac{q}{4\pi\varepsilon_0 \gamma^2} \hat{e}_z
\]

\[
\vec{E} = \vec{E}_+ + \vec{E}_- = -\frac{q}{2\pi\varepsilon_0 \gamma^2} \cdot \frac{\gamma}{\gamma} \hat{e}_y = -7.9 \times 10^9 \hat{e}_y \text{(N/C)}
\]

\[
\vec{F} = (-e) \vec{E} = (-1.60 \times 10^{-19} \text{C})(-7.9 \times 10^9 \hat{e}_y) \text{ N/C}
\]

\[
= 1.3 \times 10^{-9} \text{ N} \hat{e}_y
\]

b. Gauss’ Law. An infinite slab of thickness \( L \) has charge density \( \rho(x) = Ax \), where \( x \) is the distance to the midplane of the slab and \( A \) is a constant. What is the electric field inside the slab \((x < L/2)\)?

\[
\int \vec{E} \cdot d\vec{A} = \frac{Q}{\varepsilon_0}
\]

\[
2\beta E = \int_{-\infty}^{x} |x| dx \cdot \frac{B}{\varepsilon_0} \quad \Rightarrow \quad Q = 2\int_{0}^{x} Ax dx \cdot B
\]

\[
2B E = Ax^2 B / \varepsilon_0
\]

\[
E = \frac{Ax^2}{2\varepsilon_0}
\]
Problem 2.

a. Electric potential and work. An electric field is aligned with the \( x \)-axis and is given by \( E = 7.3x^2 \mathbf{i} \), where \( x \) is in meters and the electric field is in units of N/C. Find the electric potential difference \( V_{AB} \) between a point \( A \) at the origin and point \( B \) at position \( (x, y) = (0.2 \text{ m}, 0.4 \text{ m}) \), and give the work \( W \) done by the electric field on the point charge \( (q = 8.0 \text{ \( \mu \}) \) as it is moved from \( A \) to \( B \).

\[
V_{AB} = \int_A^B E \cdot dx = \int_0^{0.4} 7.3x^2 dx
= 1.95 \times 10^{-2} \text{ V}
\]

\[
W = q \cdot V = 8 \times 10^{-6} \text{ C} \times 1.95 \times 10^{-2} \text{ V}
= 1.56 \times 10^{-7} \text{ J}
\]

b. Electrostatic energy and capacitance. Two concentric thin conducting spheres have radii \( R_1 \) and \( R_2 \) \( (R_1 < R_2) \). A total charge \( +Q \) is placed on the inner sphere and a total charge \( -Q \) resides on the outer sphere. Find the total electrostatic energy \( U \) of this system and give the capacitance \( C \) of the two spheres.

\[
V_{A,R_2} = \int_{R_1}^{R_2} E dr = \frac{Q}{4\pi \varepsilon_0} \left( \frac{1}{R_1} - \frac{1}{R_2} \right)
\]

\[
C = \frac{Q}{V} = \frac{4\pi \varepsilon_0 R_1 R_2}{R_2 - R_1}
\]

\[
\varepsilon = \frac{1}{2} CV^2 = \frac{1}{2} \frac{4\pi \varepsilon_0 R_1 R_2}{R_2 - R_1} \left( \frac{Q^2}{4\pi \varepsilon_0 (R_2 - R_1)} \right)^2
= \frac{Q^2}{8 \pi \varepsilon_0} \frac{R_1 R_2}{R_2 - R_1}
\]
Problem 3.

a. **Motion in a magnetic field.** A charge $q = 2.0 \text{C}$ and mass $3.0 \text{ kg}$ is in uniform circular motion with speed $6.0 \text{ m/s}$, orbiting on a circle of radius $r = 4.0 \text{ m}$. It is kept in this orbit by partly by the pull of a string attached to the center of the circular path of the charge, and partly by the force from a magnetic field of strength $B = 1.0 \text{ T}$ which is perpendicular to the plane of the orbit. Find the magnitude of the force, $F$ on the charge from the string (that is, find the tension in the string).

$$\frac{mV^2}{r} = \sum_i F_i = qV\vec{B} + F = \frac{3}{4} (36) = 27 \text{ N}$$

$$qV\vec{B} = 2(6) = 12 \text{ N}$$

so $F = 15 \text{ N}$

b. **Ampère's Law for steady currents.** A long cylindrical wire of radius $R$ carries a current density $J = Cr$, where $r$ is the distance from the wire's axis, and $C$ is a constant. What is the magnitude of the magnetic field, $B$, at a point inside the wire ($r < R$)?

$$\oint B \cdot dl = \mu_0 I = \mu_0 \int_0^1 J \cdot dA$$

$$B2\pi r = \mu_0 \int_0^r Cr'^2 2\pi r' dr' = \mu_0 C \frac{2\pi r^3}{3}$$

$$B = \frac{\mu_0 Cr^2}{3}$$
Problem 4.

a. Faraday's Law and Induction. A long straight wire lies in the plane of a square wire loop (length $a$ to a side), as shown. The distance between the straight wire and the nearest edge of the square loop is $a$. Find the mutual inductance $M$ of this pair of wires, and give the electromotive force $\mathcal{E}$ induced in the square loop if the current in the straight wire varies as $I = Bt$, where $B$ is a constant.

Choose $x$-axis as shown in figure.

Using Faraday's law

$$\oint B \cdot dl = \omega_0 I$$

$$\frac{\partial}{\partial t} \int B \cdot dA = \omega_0 I$$

$$2\pi r B = \omega_0 I$$

$$B = \frac{\omega_0 I}{2\pi r}$$

The flux in the square wire is:

$$\Phi_B = B A = \frac{\omega_0 I}{2\pi} \int |dx| = \frac{\omega_0 I a \ln 2}{2\pi}$$

b. A Circuit. The components in the circuit in the diagram below are initially uncharged and no current is flowing. In this circuit $R_1 = R_2 = 2 \, \Omega$, $C = 4 \, \text{mF}$, $L = 6 \, \mu\text{H}$, and $\mathcal{E} = 6 \, \text{V}$. At time $t = 0$ s the switch is closed. Give the total energy $U$ stored in the electric field (inside the capacitor) and magnetic field (inside the inductor) at a large time after the switch is closed.

When $t \to \infty$

$$I = \frac{\mathcal{E}}{R_1 + R_2} = \frac{6 \, \text{V}}{2 \, \Omega} = 3 \, \text{A}$$

$$V_c = IR_2 = 3 \, \text{V}$$

$$U_c = \frac{1}{2} C V_c^2 = \frac{1}{2} \times 4 \, \text{mF} \times 3^2 \, \text{V} = 18 \, \text{mJ}$$

$$U_L = \frac{1}{2} L I^2 = \frac{1}{2} \times 6 \, \mu\text{H} \times 15^2 \, \text{A}^2 = 675 \, \text{mJ}$$

$$U_{total} = U_c + U_L = 18.75 \, \text{mJ}$$

c. An AC Device. What value of resistance $R$, capacitance $C$, and inductance $L$ would you choose if you wanted to make a device (a series RLC circuit, as shown) that draws an average of 100 W of power from a 120 Vrms, 60 Hz wall socket? (There is no unique answer to this problem.)

$$X_C = \frac{1}{\omega C}$$

$$X_L = \omega L$$

$$\phi = \arctan \left( \frac{V_{rms} I_{rms}}{Z} \right)$$

$$< p > = V_{rms} I_{rms} \cos \phi$$

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

$$\omega L = \frac{V_{rms}}{X_C}$$

$$\omega = 2\pi f = 344$$

$$R = 144 \, \Omega$$

$$L = \frac{1}{\omega^2 C F}$$

$$\omega = 1183 \, \text{MHz}$$

$$C = \frac{1}{\omega^2 L F}$$
Problem 5.

**a. Electromagnetic radiation.** An electromagnetic plane wave propagates in the positive $y$ direction with a frequency of $88.3$ MHz and an average intensity of $\bar{S} = 0.5$ W/m$^2$. If the electric field vector $\vec{E}$ oscillates in the $x$ direction, write an expression for $\vec{E}$ and the magnetic field vector $\vec{B}$, as functions of space and time.

\[
\vec{E} = +E_p \sin \left( k y - \omega t \right) \hat{x}\quad \quad \quad \quad \quad \vec{B} = -B_p \sin \left( k y - \omega t \right) \hat{k}
\]

\[
\bar{S} = \frac{E_p^2}{2\mu_0 c} \quad \Rightarrow \quad E_p = \sqrt{2\mu_0 c \bar{S}} \quad \Rightarrow \quad B_p = \sqrt{\frac{2\mu_0 \bar{S}}{c}} = 6.64 \times 10^{-8} \text{ T}
\]

\[
\omega = 2\pi f = 5.55 \times 10^8 \text{ s}^{-1}
\]

\[
k = \frac{\omega}{c} = 1.85 \text{ m}^{-1}
\]

**b. Energy and momentum of light.** A small robotic airplane has an area of $3$ m$^2$ as seen from below. Its lower surface absorbs all the energy beamed up to it by a powerful solar collecting mirror. If the intensity of the light hitting the underside of the plane from the mirror is $1000$ kW/m$^2$, find the total power $P$ transferred to the airplane by the light, and give the upward force $F_{\text{up}}$ from radiation pressure.

\[
P = \bar{S}A = 3 \times 10^6 \text{ W}
\]

\[
p = \frac{\bar{S}}{c} \quad \quad \quad \quad F = pA
\]

\[
F = \frac{\bar{S}A}{c} = 0.01 \text{ N}
\]
Problem 6.

a. Refraction. The image of a fish in a calm pond appears to be at a distance of 1 meter away from the surface as viewed by someone peering down into the pond. If the index of refraction of the pond water is \(n = 1.33\), what is the actual distance \(d\) between the fish and the surface?

\[
L = d \tan \theta \quad \text{and} \quad L = 1m \cdot \tan \theta \quad \Rightarrow \quad d = \frac{\tan \theta}{\tan \theta} = \frac{\theta}{\theta} = 1m \cdot \tan \theta + 1\'
\]

for \(\alpha = 30^\circ\), \(1.33 \sin 30^\circ = 1.0 \sin 30^\circ\), \(\alpha' = 4.68^\circ\)

\[
S_0: \quad d = \frac{\tan 4.68^\circ}{\tan 30^\circ} = 1.53 \text{ m} \quad \text{d is slightly different for different } \alpha.
\]

Or:

\[
\frac{n}{n'} + \frac{n'}{L'} = \frac{n-n_1}{R}, \quad \text{where } n' = 1.0, \quad n = 1.33, \quad L' = -1m, \quad R = \infty, \quad L = d
\]

\[
\Rightarrow \quad d = l = 1.33 \text{ m} + 1'
\]

b. An optical device. Complete the diagram below showing an object being imaged by a concave lens (focal length \(|f| = 1 \text{ m}|\)). Specifically, calculate the location of the image and its height, and draw the image to scale in the diagram.

\[
\begin{align*}
+1' & \quad \frac{1}{f} + \frac{1}{f'} = \frac{1}{f} \quad l=2, \quad f=-1 \\
+1' & \quad \frac{1}{f} = \frac{1}{l'} \quad \frac{1}{l'} = \frac{1}{l} \\
+1' & \quad l' = -\frac{2}{3} \text{ m}
\end{align*}
\]

\[
M = -\frac{l'}{l} = \frac{h'}{h} \\
+1' \quad h' = -\frac{0.75}{2} = \frac{3}{2} \text{ m}
\]

c. Interference. A single power supply drives two simple radio antennae so that the radio waves are of equal intensity and have the same phase at the antennae. Each antenna broadcasts an equal amount of power in all directions in the horizontal plane. If the antennae are spaced 1.25 wavelengths apart, specify the number of directions in the horizontal plane along which the combined signal will be a maximum.

\[
\begin{align*}
+1' & \quad d \sin \theta = m \lambda \\
& \quad \sin \theta = \frac{m \lambda}{1.25 \lambda} = \frac{m}{1.25} \quad (m = 0, \pm 1, \pm 2, \ldots)
\end{align*}
\]

\[
+1' \quad \text{since } \sin \theta \leq 1, \quad m/1.25 \leq 1, \quad m = 0, \pm 1
\]

\[
\begin{align*}
& \quad m = 0, \quad \theta = 0 \\
& \quad m = 1, \quad \theta = \sin^{-1} \frac{1}{1.25} = 53.13^\circ \quad \text{In the whole plane,} \\
& \quad m = -1, \quad \theta = -53.13^\circ \\
& \quad 6 \text{ directions}
\end{align*}
\]