Problem 1: Electric field in vacuum

Electric potential is given \( \phi(x, y, z) = ax^2 + by^2 - c^2 \)

The electric field is given by

\[
\vec{E} = - \nabla \phi(x, y, z) = -(\partial_x x^2 + \partial_y y^2 + \partial_z z^2)(ax^2 + by^2 - c^2)
\]

\[
\vec{E} = -2(axx + byy - czz)
\]

The charge density is found from Poisson's Equation,

\( \nabla^2 \phi = - \rho/\varepsilon_0 \)

\[
\Rightarrow \rho(x, y, z) = - \varepsilon_0 \nabla^2 \phi = -\varepsilon_0 \left( 2x^2 \phi + 2y^2 \phi + \partial_z z^2 \phi \right)
\]

\[
\rho(x, y, z) = -2\varepsilon_0 (a + b - c)
\]
Problem 2: Polarization of Dielectrics

a) The electric field can be found by exploiting the planar symmetry of the setup:
\[ \oint \mathbf{d} \mathbf{A} \cdot \mathbf{E} = Q_{\text{free, enc}} \quad \text{and} \quad \mathbf{D} = \mathbf{E} \epsilon \]

Here, \( Q_{\text{free, enc}} = \rho \) which is the charge that is placed on the slab.

By symmetry, the displacement (as well as the electric) field can only point in the \( \pm z \)-direction.

\[ \begin{align*}
\text{Inside} & \quad |z| < a \\
\mathbf{D} \cdot \mathbf{A} = \rho \mathbf{A} \cdot \mathbf{A} & \quad \Rightarrow \quad \mathbf{D} = \rho z \mathbf{\hat{z}} \\
\Rightarrow \quad \mathbf{E} &= \frac{\rho z}{\epsilon} \mathbf{\hat{z}} \\
\text{Outside} & \quad |z| > a \\
\mathbf{D} \cdot \mathbf{A} = \rho \mathbf{A} \cdot \mathbf{A} & \quad \Rightarrow \quad \mathbf{D} = \pm \rho a \mathbf{\hat{z}} \\
\Rightarrow \quad \mathbf{E} &= \pm \frac{\rho a}{\epsilon_0} \mathbf{\hat{z}}
\end{align*} \]

\[ \tilde{E}(z) = \pm \begin{cases} 
\frac{f|z|}{\epsilon} & |z| < a \\
\frac{\rho}{\epsilon_0} & |z| > a
\end{cases} \]

with \( f = \pm 1 \) if \( z > a \)
b) The bound charges are \( \rho_b = -\nabla \cdot \hat{P} \) and \( \sigma_b = \hat{P} \cdot \hat{n} \). So first we need the polarization

\[
\hat{P} = \varepsilon_0 \chi_0 \hat{E} = \varepsilon_0 \left( \frac{\varepsilon}{\varepsilon_0} - 1 \right) \hat{E}
\]

\[
\hat{P} = \begin{cases} 
\varepsilon \frac{\varepsilon - \varepsilon_0}{\varepsilon} \hat{E} & \text{if } |z| < a \\
0 & \text{if } |z| > a
\end{cases}
\]

The volume bound charge is

\[
\rho_b = -\nabla \cdot \hat{P} = -(\partial_x \hat{x} + \partial_y \hat{y} + \partial_z \hat{z}) \left( \varepsilon - \varepsilon_0 \right) \frac{\rho}{\varepsilon}
\]

\[
\rho_b = -\left( \varepsilon - \varepsilon_0 \right) \frac{\rho}{\varepsilon}
\]

The surface bound charges are:

\[ z = a \] \[ \sigma_b(a) = \hat{P} \cdot \hat{n} = \left( \varepsilon - \varepsilon_0 \right) \frac{\rho a}{\varepsilon} \]

\[ z = -a \] \[ \sigma_b(-a) = \hat{P} \cdot (-\hat{n}) \bigg|_{-a} = -\left( \varepsilon - \varepsilon_0 \right) \frac{\rho (-a)}{\varepsilon} \]

\[ \Rightarrow \sigma_b(\pm a) = \left( \varepsilon - \varepsilon_0 \right) \frac{\rho a}{\varepsilon} \]
Problem 3: Lorentz Force

Only those particles with a certain speed \( v_0 \) will be undeflected in the speed filter, and can exit out the other side.

Particles travelling too fast \((v > v_0)\) will be deflected upwards \((-\hat{y}\text{-direction})\) by the magnetic field.

Particles travelling too slow \((v < v_0)\) will be deflected downwards \((\hat{y}\text{-direction})\) by the electric field.

The speed at which particles are not deflected is the speed at which the Lorentz force vanishes

\[
F_L = q \left( \vec{E} + \vec{v}_0 \times \vec{B} \right) = q \left( \vec{E} - v_0 \vec{B} \right) = 0 \]

\[\Rightarrow \quad v_0 = \frac{E}{B}\]
Problem 4: Basic Properties of Magnetic Fields

The $z$-component of an axially symmetric magnetic field is given by $B_z(t) = B_0 (1 + \frac{t}{t_0})$.

The general form of the field is $\vec{B} = Br \hat{r} + B\phi \hat{\phi} + B_z \hat{z}$ where $B\phi = \text{const}$ for an axially symmetric field.

The divergence of $\vec{B}$ must vanish:

$$\nabla \cdot \vec{B} = \frac{1}{r} \frac{\partial}{\partial r} (r Br) + \frac{\partial}{\partial z} B_z = 0$$

$$= \frac{1}{r} Br + \frac{\partial r Br}{\partial r} + \frac{B_0}{t_0} = 0$$

There are two ways to solve this differential equation:

1. The first is to recognize the solution is of the form $Br(r) = A_1 r + A_2$, but $A_2 = 0$ since physically we can't let the field diverge at $r = 0$. Then putting this into the DE:

$$\frac{1}{r} (A_1 r) + A_1 + \frac{B_0}{t_0} = 0 \Rightarrow A_1 = -\frac{B_0}{2t_0}$$

$$\Rightarrow Br(r) = -\frac{B_0 r}{2t_0}$$

2. The second way is to rewrite the DEs and integrate:

$$\overbrace{Br + r \frac{dr Br}{dr}} = -\frac{B_0}{t_0} r$$

$$= \frac{\partial}{\partial r} (r Br) = \frac{\partial r}{\partial r} (r Br)$$

$$\Rightarrow d (r Br) = -\frac{B_0}{t_0} r \, dr$$

Since $Br = Br(r)$, the partial derivatives become full derivatives.

Therefore, $Br = Br(r)$.
Integrating gives

\[ r B_r = -\frac{B_0 r^2}{2h_0} + C \]

\[ B_r = -\frac{B_0 r}{2h_0} + \frac{C}{r} \]

where \( C = 0 \) again because we cannot have \( B \) diverge at \( r = 0 \).

\[ \Rightarrow B_r(r) = -\frac{B_0 r}{2h_0} \]

The angle \( \alpha \) is given by

\[ \tan \alpha = \frac{B_r}{B_z} \bigg|_{r=R, z=h} = -\frac{B_0 R}{2L_0} \frac{1}{B_0 \left(1 + \frac{h}{h_0}\right)} \]

\[ \alpha = \arctan \left( -\frac{R}{2(h_0 + h)} \right) \]

So at \( R = 0 \), \( \alpha = 0 \)

at \( R \to \infty \), \( \alpha = 90^\circ \)