Problem 1: Classical Electron radius

Denote the electron radius as $R$ and total charge as $e$. We approximate it as a uniformly charged sphere with charge density $\rho = \frac{e}{\frac{4}{3} \pi R^3}$.

We want to find its electrostatic energy in three different ways.

e) Assemble the sphere layer by layer.

Consider a partially assembled sphere of radius $r < R$ and charge $q < e$. The potential due to this sphere at the surface of it is

$$\Psi(r) = \frac{1}{4\pi \varepsilon_0} \frac{qe}{r} = \frac{1}{4\pi \varepsilon_0} \frac{e\rho^2}{r^3}$$

Now, suppose we want to add a thin shell of thickness $dr$ and charge $dq$ to the partially assembled sphere. The charge $dq$ can be expressed as

$$dq = 4\pi r^2 dr \rho = 3e \frac{r^2 dr}{R^3}$$

The work required to add the shell is

$$dW = dq \Psi(r) = 3e \frac{r^2 dr}{R^3} \frac{1}{4\pi \varepsilon_0} \frac{e\rho^2}{R^3} = \frac{3e^2 r^4}{4\pi \varepsilon_0 R^3} dr$$
We can repeat this process and add up (i.e. integrate) all of the work required to assemble the sphere.

\[ W = \frac{3e^2}{4\pi \varepsilon_0 R^2} \int_0^R dr \, r^n = \frac{1}{4\pi \varepsilon_0} \frac{3e^2}{cR} = W \]

b) Use the electric field. \[ W = \frac{\varepsilon_0}{2} \int_{\text{all space}} dV \, E^2 \]

Since we need to integrate over all space, we need to know the electric field everywhere:

\[ \hat{E}(r) = \frac{e^2}{4\pi \varepsilon_0} \begin{cases} \frac{1}{r^2} & r < R \\ \frac{1}{R^2} & r > R \end{cases} \]

So, the energy is

\[ W = \frac{\varepsilon_0}{2} \left( \frac{e^2}{4\pi \varepsilon_0} \right)^2 \int d\Phi \int dS \sin \phi [ \int_0^R dr \, r^2 \left( \frac{1}{r^2} \right)^2 + \int_{R}^{\infty} dr \, r^2 \left( \frac{1}{r^2} \right)^2 ] \]

\[ = \frac{1}{2} \frac{e^2}{4\pi \varepsilon_0} \left[ \frac{r^5}{5R^6} \bigg|_0^R - \frac{1}{r} \bigg|_0^\infty \right] \]

\[ = \frac{1}{2} \frac{e^2}{4\pi \varepsilon_0} \left[ \frac{1}{5R^6} + \frac{1}{R^2} \right] = \frac{1}{4\pi \varepsilon_0} \frac{3e^2}{cR} = W \]
c) Use the potential \( W = \frac{1}{2} \int dV \rho(r) \psi(r) \)

Here, \( \rho = \frac{e}{\frac{4}{3} \pi R^3} = \text{const} \) and we need to integrate \( \psi(r) \) over the volume of the sphere. (since \( \rho = 0 \) outside)

First, we need to calculate the potential:

\[
\psi(r) = -\int_0^R \frac{dV}{V} \L E = -\int_0^R \frac{dV}{V} E_{\text{outside}} - \int_0^R \frac{dV}{V} E_{\text{inside}} = \frac{e}{4\pi \varepsilon_0} \left[ -\int_0^R \frac{dV}{V} \right] \\
= \frac{e}{4\pi \varepsilon_0} \left( \frac{1}{R} - \frac{r^2 - R^2}{2R^3} \right) = \frac{e}{4\pi \varepsilon_0} \frac{1}{2} \left( \frac{3}{R^2} - \frac{r^2}{R^3} \right)
\]

So then the work is

\[
W = \frac{1}{2} \frac{e}{\frac{4}{3} \pi R^3} \int_0^R \frac{dV}{V} r^2 \frac{e}{4\pi \varepsilon_0} \frac{1}{2} \left( \frac{3}{R^2} - \frac{r^2}{R^3} \right)
\]

Angular integral

\[
W = \frac{3e^2}{4\pi \varepsilon_0} \frac{1}{4R^3} \left( \frac{R^2}{12} - \frac{R^5}{15} \right)
\]

\[
W = \frac{1}{4\pi \varepsilon_0} \frac{3e^2}{5R}
\]

d) Numerical value for electron radius

To estimate the electron radius, assume that the rest energy is equal to the electrostatic energy:

\[
\frac{1}{4\pi \varepsilon_0} \frac{3e^2}{5R} = MeC^2.
\]
Solving for the radius:

\[ R = \frac{3e^2}{4\pi\varepsilon_0} \frac{1}{\frac{5mc^2}{3}} \]

\[ = \frac{3(1.6 \times 10^{-19} \text{C})^2}{4\pi (8.85 \times 10^{-12} \text{C}^2/\text{Nm}^2)} \frac{1}{5(9.1 \times 10^{-31} \text{kg})(3 \times 10^8 \text{m/s})^2} \]

\[ R \approx 1.7 \times 10^{-15} \text{m} \]

What if we had decided to model the electron as a spherical shell rather than a solid sphere? The electrostatic energy would have been smaller. This can be seen by writing the electric field of a shell:

\[ E(r) = \begin{cases} \frac{1}{\frac{4\pi\varepsilon_0}{r^2}}, & r > R \\ \text{and} & \\
0, & r \leq R \end{cases} \]

Since the energy is the integral of the square of the field, the first term in (b) vanishes.
Problem 2. Electrostatic energy of spherical shell with charge density \( \sigma(r, \theta) = \sigma_0 \cos \theta \).

We want to calculate the electrostatic energy in two ways: using the electric field and using the potential. So first, we need to know both of those things.

The electric potential of a spherical shell (which you know either from class or from example 3.9 in Griffiths) is

\[
U(r, \theta) = \begin{cases} \frac{\sigma_0}{3 \varepsilon_0} \frac{r \cos \theta}{r} & r < R \\ \frac{\sigma_0 R^3}{3 \varepsilon_0} \frac{\cos \theta}{r^2} & r > R \end{cases}
\]

The electric field can be found by calculating the gradient:

\[
E(r, \theta) = -\nabla U(r, \theta) = \left( \frac{\sigma_0}{3 \varepsilon_0} \frac{\cos \theta}{r} \right) \hat{r} + \left( \frac{\sigma_0 R^3}{3 \varepsilon_0} \frac{\cos \theta}{r^2} \right) \hat{\theta}
\]

\[
= \begin{cases} \frac{\sigma_0}{3 \varepsilon_0} \left( -\cos \theta \hat{r} + \frac{1}{r} \sin \theta \hat{\theta} \right) & r < R \\ \frac{\sigma_0 R^3}{3 \varepsilon_0} \frac{1}{r^2} \left( 2 \cos \theta \hat{r} + \sin \theta \hat{\theta} \right) & r > R \end{cases}
\]

a) Energy using the electric field \( W = \frac{\varepsilon_0}{2} \int_{\text{All Space}} \text{d}V E^2 \).

Here we actually need the square of the field:

\[
E^2(r, \theta) = E \cdot E = \frac{\sigma_0^2}{9 \varepsilon_0^2} \begin{cases} \frac{1}{r} \cdot \frac{R}{r} & r < R \\ \frac{R^6}{r^6} \left( 4 \cos^2 \theta + \sin^2 \theta \right) & r > R \end{cases}
\]
Inside: \( W_{\text{in}} = \frac{\varepsilon_0}{2} \frac{\sigma^2}{\varepsilon_0} \frac{4}{3} \pi R^3 \)

Outside:

\[
W_{\text{out}} = \frac{\varepsilon_0}{2} \frac{\sigma^2}{\varepsilon_0} 2\pi \int_0^{\pi} d\theta \sin \theta (4 \cos^2 \theta + \sin^2 \theta) \int_0^R dr \frac{r^2 R^6}{r^6} = \frac{\varepsilon_0}{2} \frac{\sigma^2}{\varepsilon_0} 2\pi \frac{4}{3} R^3 = \frac{R^3}{3}
\]

Total: \( W = W_{\text{in}} + W_{\text{out}} = \frac{2\pi}{9\varepsilon_0} \sigma^2 R^3 = W \)

b) Using the potential

\[
W = \frac{1}{2} \int \text{d}V \rho \varphi = \frac{1}{2} \int \text{d}a \sigma \varphi \quad \text{with} \quad \text{d}a = R^2 \sin \theta d\theta d\phi
\]

Since we are dealing with a surface charge, we want the potential at \( r=R \):

\[
\varphi (r, \theta) = \frac{\sigma}{3 \varepsilon_0} R \cos \theta
\]

So the energy is

\[
W = \frac{1}{2} \int R^2 \sin \theta d\theta d\phi \sigma \cos \theta \frac{\sigma}{3 \varepsilon_0} R \cos \theta
\]

\[
= \frac{1}{2} R^3 \frac{\sigma^2}{3 \varepsilon_0} 2\pi \int_0^{\pi} d\theta \sin \theta \cos \theta \cos^2 \theta
\]

\[
= \frac{2}{3}
\]

\[
W = \frac{2\pi}{9\varepsilon_0} \sigma^2 R^3
\]