Problem 1

Ampere's Law: \( \oint \mathbf{d}\mathbf{l} \cdot \mathbf{B} = \mu_0 I_{\text{enc}} \)

Note: "Thin" means infinitely extended in the other direction.

- Since magnetic field is perpendicular to current, there is no field in \( x \).
- By symmetry there is no field in \( y \).
- At \( y = 0 \) we have \( \mathbf{B} = 0 \) due to symmetry.

Ampere's law gives:

\[
B \cdot 2L = \mu_0 I A = \mu_0 J (2yL) \\
\Rightarrow B = \mu_0 J y \\
\Rightarrow B = \frac{\mu_0 J y}{2} \quad \text{for} \quad -\frac{d}{2} < y < \frac{d}{2}
\]
Problem 2

a) Since electrons are free to move in conductors, the rotation of the cylinder pushes them outwards, with a force with magnitude \( m\omega^2 r \), leaving a positive "core" in the cylinder. This positive core exerts an attractive electric force of magnitude \( eE \) on the electrons.

- Note that electrons at the edge of the cylinder have speed \( v = \omega R = 500 \text{ m/s} \ll c \), so we can neglect relativistic effects.
- So the electric field is \( eE = m\omega^2 r \Rightarrow \vec{E} = \frac{m\omega^2 r}{e} \).

b) In order for there to be no electric field, we need the Lorentz force to be zero:

\[
F_L = e(E + vB) = e(E + \omega rB) = 0
\]

\[
\Rightarrow |B| = \frac{|E|}{\omega r} = \frac{m\omega}{e} = \frac{(9.1 \times 10^{-31} \text{ kg})(1000 \text{ rad/s})}{1.6 \times 10^{-19} \text{ C}}
\]

\[
|B| = \frac{m\omega}{e} = 5.7 \times 10^{-9} \text{ T}
\]
Problem 3

\[ \vec{A}_1 = \frac{B}{2} (-y, x, 0) \quad \vec{A}_2 = B (0, x, 0) \]

Magnetic field is \( \vec{B} = \vec{\nabla} \times \vec{A} \).

\[ \vec{B}_1 = \vec{\nabla} \times \vec{A}_1 = \frac{B}{2} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \\ 0 & 0 & \frac{\partial}{\partial x} \end{vmatrix} = \frac{B}{2} \left[ \frac{2π}{(\frac{\partial}{\partial x})(-\frac{\partial}{\partial y})} + \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right)^2 \right] \]

\[ = B \bar{z} \]

\[ \vec{B}_2 = \vec{\nabla} \times \vec{A}_2 = B \left( \frac{2}{\partial x} x + \frac{2}{\partial y} y - \frac{2}{\partial x} \right) \times (x \hat{y}) = B \frac{2x}{\partial x} \hat{z} = B \vec{z} \]

\[ \Rightarrow \vec{B}_2 = \vec{B}_1 = B \vec{z} \]

Since \( \vec{B} = \vec{\nabla} \times \vec{A} \), we can add the gradient of any scalar function \( \vec{\nabla} f \) to \( \vec{A} \) without changing \( \vec{B} \). This is because the curl of a gradient is always zero, i.e., \( \vec{\nabla} \times \vec{\nabla} f = 0 \).

\( \Rightarrow \vec{A}_1 \) and \( \vec{A}_2 \) can only differ up to an added gradient.

We can see this in the present case if we find such an \( f \):

\[ \vec{A}_2 = \vec{A}_1 + \vec{\nabla} f \]

\[ \Rightarrow \vec{\nabla} f = \vec{A}_2 - \vec{A}_1 = B (0, x, 0) - B (-\frac{y}{2}, \frac{x}{2}, 0) = B \left( \frac{y}{2}, \frac{x}{2}, 0 \right) \]

\[ = \left( \frac{x}{\partial x}, \frac{y}{\partial y}, \frac{z}{\partial z} \right) \]

It can be seen easily that \( f = \frac{B y x}{2} \) satisfies the condition.

\[ \vec{A}_1 = \frac{B}{2} (-y, x, 0) \Rightarrow \text{Symmetric gauge} \]

\[ \vec{A}_2 = B (0, x, 0) \Rightarrow \text{Landau gauge} \]

A possible alternate form for \( \vec{A} \) is \( \vec{A}_3 = B (-y, 0, 0) \).