Lectures 21+22+23

Electric current

a) If we create an electric field in a conductor, it will result in positive charges flow along the field and negative ones opposite to it. (If the charges one sign cannot move for some reason only the other kind will contribute.) Such organized motion of charges is called electric current.

F. c. is characterized by current intensity: if charge $\Delta Q$ crosses a particular cross section of a conductor in time $\Delta t$, we say there is a current of $I = \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt}$ flowing through it.

Units: $[I] = \frac{C}{T} = A$ (Ampere) or "amp" for brevity.

Convention: we take the direction of the flow of positive charges as the current direction. Negative charges flowing in the opposite direction create the same current as far as the direction is concerned.

$\Theta$ or $\ominus$ or $\overline{\Theta}$ or $\overline{\ominus}$ or $\overline{1} = \frac{dQ^+}{dt} + \frac{dQ^-}{dt} = \frac{dQ^+}{dt} - \frac{dQ^-}{dt}$
Example: transient current in an electrically disconnected conductor.

Turn on $E_o$ to $+T$ and $-E_o$ to $-T$. For the case of a surface charge, the current density is $\frac{dQ}{dt} = \frac{d}{dt} \int J \cdot dS = \frac{d}{dt} \int E \cdot dA$.

### Notes:

- Current carriers do not see any field force.
- The absence of a field in this case is ensured.
- The presence of an external electric field.
- The current is due to the drift of electrons.
- The resulting electric field is due to the charges of the conductor.
b) **Current Density**

What if the charge flow is non-uniform through a particular cross section of a conductor?

We can introduce **current density** in this case, \( \vec{j} \).

It is a vector quantity.

It is constructed in the following way: Introduce the direction of current carriers at point \( \vec{A} \) — this is the direction of \( \vec{j} \). To assign a magnitude to \( \vec{j} \), draw a surface element \( \Delta \vec{S} \) (with \( \Delta S \to 0 \)) and calculate the current through it, \( \Delta I \). Then

\[
\vec{j} = \frac{\Delta I}{\Delta S}
\]

**Corollary 1:** For an arbitrary surface element,

we get

\[
\Delta I = \vec{j} \cdot \Delta \vec{S} = \vec{j} \cdot \hat{n} \Delta S
\]

**Corollary 2:** For the current through a finite surface,

we get

\[
I = \int \vec{j} \cdot \hat{n} \, dS
\]
c) In general, charge carriers participate in two motions: diffusion and organized travel:

\[ \langle \vec{V}_d(t) \rangle = 0, \quad \langle \vec{V}_o(t) \rangle \neq 0. \]

Thus, \( \langle \vec{V}(t) \rangle_{\text{diff}} = \langle \vec{V}_d(t) \rangle_{\text{diff}}. \)

By its definition, \( \vec{j} \) points along \( \langle \vec{V} \rangle = \langle \vec{V}_o \rangle. \)

In fact, we can relate \( \vec{j} \) to \( \langle \vec{V}_o \rangle \) in a system of charge carriers with density number density (concentration) \( n_e \) and charge \( q \):

\[ \Delta \Sigma = \frac{\Delta q}{\Delta t} = \Delta s \cdot \vec{j}_n = \vec{n} \cdot \vec{v} \cdot \Delta t \cdot \Delta s \text{ - number of particles.} \]

Thus:

\[ \vec{j} = q n_e \cdot \langle \vec{V} \rangle \]

Note that only \( q \cdot \vec{v} \) enters, thus if an electric field pulls positive charges to the right (\( n_+ > 0 \)) and negative ones to the left (\( n_- < 0 \)), the corresponding products \( q^+ n_+ \) and \( q^- n_- \) have the same sign. In general:

\[ \vec{j} = q^+ n^+ \vec{v}^+ + q^- n^- \vec{v}^- \]
John's Law

Having discussed the concepts of current and current density, we need to relate them to the electric field in conductors. There are two main questions in this regard:

1) How to create an electric field in a conductor?

2) Why an electric field, once created, does not cause an indefinite acceleration of charges.

To answer the first question, recall that the field inside conductors was zero due to charges on their surfaces created in the process of screening the external field. Thus, if one wants a finite field inside a conductor, he must remove the charges from the surfaces all the time – that is, allow a current to pass through a conductor.

If \( \Phi_I > \Phi_R \), a current will flow to the right (see Fig.) and to keep it steady, the "free" charge built up on the right end needs to be passed back onto the left end.
Notice that this would simply go against the potential difference outside the conductor, as \( \Delta \times \vec{E} = 0 \). This means we need an extrinsic "push" to remove the positive charge from the right end, and supply it to the left end. This is what a battery does—by using fuels of chemical origin, we will talk about it later.

Now, what is the relation between the current density and electric field? To answer this question, we have to keep in mind that conductors are never perfect and there are imperfections that scatter electrons. Thus, when an \( \vec{E} \)-field is applied, there are two things that happen to charge carriers: they are accelerated by the electric field, and they lose momentum because of collisions with imperfections, phonons, each other, etc. All these factors contribute to a sort of a "friction-like" acting on electrons,

\[
\frac{d\vec{v}}{dt} = q\vec{E} - \frac{\vec{v}}{\tau}, \quad \tau \text{ — mean scattering time,}
\]

momentum loss rate due to "friction".

We see that in steady state, \( \vec{v} \to 0 \), we have

\[
q\vec{E} = \frac{\vec{v}_0}{\tau}, \quad \text{or} \quad \left| \vec{v} \right| = \frac{q\vec{E}}{m} \quad \text{— drift velocity}
\]
Plugging this back into the expression for \( \vec{j} \) we get
\[
\vec{j} = nq \vec{v} = \frac{nq^2 \tau}{m} \vec{E}, \quad \text{or} \quad \vec{j} \propto \vec{E}.
\]

More general statement,
\[
\left[ \vec{j} = \sigma \vec{E} \right] \quad - \quad \text{Ohm's law in differential form.}
\]
\( \sigma \) - conductivity

\[
\sigma = \frac{nq^2 \tau}{m} \quad \text{in the model we just considered ("drude model")}
\]

It is quite idealized but has all the right features: \( \sigma \) grows with increasing \( n \) - density of carriers, and scattering time - the less you scatter, the better you conduct.

We can also write
\[
\vec{E} = \frac{1}{\sigma} \vec{j} = \frac{1}{\rho} \vec{j}, \quad \sigma \text{- resistivity;}
\]
\[
\rho = \frac{1}{\sigma}.
\]

The fact that \( \vec{j} \) and \( \vec{E} \) are proportional to each other implies that the integral characteristics, current \( I \) and voltage \( V \) across a conductor are also proportional:
\[
I = GV \quad \text{or} \quad V = RI \quad \text{- Ohm's law.}
\]
\( G \) - conductance, \( R \) - resistance.
Units:

For integral characteristics:
\[ [R] = \left[ \frac{V}{A} \right] = \frac{\text{Volt}}{\text{ampere}} = \frac{V}{A} = \text{Ohm}. \]

\[ [G] = \frac{1}{\text{Ohm}} = \text{S} \text{ (Siemens)} \text{, } \text{"inverse Ohm"}. \]

For differential characteristics:
\[ [\sigma] = \left[ \frac{E}{J} \right] = \left[ \frac{\text{Volts}}{\text{Coulombs} \text{/m}^2 \text{sec}^{-1}} \right] = \left[ \text{L} \cdot \frac{V}{A} \right] = \text{m. ohm}. \]

\[ [\varepsilon] = \left[ \frac{1}{\sigma} \right] = \frac{1}{\text{m. ohm}}. \]

Example:

\[ \rho_{\text{Cu}} = 1.7 \cdot 10^{-8} \text{ Ohm} \cdot \text{m} \]

\[ \rho_{\text{Fe}} = 2.5 \cdot 10^5 \text{ Ohm} \cdot \text{m} \]

13 orders of magnitude difference!

We see that all materials conduct to some extent. Can we get a criterion that would separate the conductors and insulators qualitatively? The temperature dependence of \( \rho \) can be such a thing:
Conductor: lots of free carriers even at \( T \to 0 \), \( \Rightarrow \) 

\( T \) only adds to scattering \( \Rightarrow \) 

\( \tau(T) \downarrow \Rightarrow \sigma(T) \uparrow \Rightarrow \rho(T) \downarrow \)

\( \left( \frac{\sigma \propto \hbar q^2 T}{m} \right) \) qualitatively.

Insulators: no free charges at \( T \to 0 \) so the conductor 

\( \tau \) is due \( \Rightarrow \) \( \tau(T) \downarrow \Rightarrow \sigma(T) \uparrow \).

Special class of materials: superconductors.

Roughly, \( \sigma(T) \) drops to zero (unmeasurable) at some \( T \geq 0 \). 

"Critical temperature." Highly quantum mechanical state.

NB: Superconductors is NOT the same as "ideal metal" that has no superiromes. If you're interested in this read about "Meissner effect."
e) Relation between differential and integral characteristics of a conductor.

The typical experimental setup is shown in the fig. Ohm's law tells us that

\[ V = V_L - V_R = I \cdot R \]

where \( V_L \) and \( V_R \) are determined by the electrodes placed onto the conductor in question.

(\textit{electrode} = piece of metal in contact with the system in this case)

What is the relation between \( R \) and \( S \)? In general, very hard question for oddly shaped conductors, so \( R \) is determined experimentally.

But there is a practically important simple case of a wire of length \( L \) and cross-sectional area \( S \), with constant cross section along the wire;

\[ V = V_L - V_R = I \cdot R \]

defining the left and right sides of which are kept at potentials \( V_L \) and \( V_R \).
We will show in a minute that the field is uniform in this wire, but knowing this we get \( \Phi_L > \Phi_R \Rightarrow F \text{ points to the right} \)

\[
E = \rho j \Rightarrow \frac{\Phi_L - \Phi_R}{L} = \rho \cdot \frac{I}{S} \Rightarrow \frac{V}{L} = \frac{\rho \cdot I}{S} \Rightarrow \boxed{V = \frac{\rho \cdot I}{S}}
\]

\( (\text{s is uniform and } j = \frac{I}{S}) \)

or \[
\boxed{R = \frac{\rho \cdot L}{S}}
\]

This is a very sensible result: the longer the wire, the higher its resistance; the wider the wire, the smaller its resistance as there are more space for the current to flow through.

Example: resistance of a hollow cylinder

In this case the current is driven radially, the inner and outer surfaces being kept at potentials \( V_i \) and \( V_o \), respectively.

Note that for \( b - a < 2a \) (thin cylinder), the cylinder looks like film with thickness of \( b - a \), and cross section area of \( L \cdot 2\pi a \).
So in this limit we expect
\[ R \left( \frac{b-a}{a} \leq 1 \right) = \frac{b-a}{L \cdot 2\pi a} \cdot \]

The field inside the material is created by the charges on its surfaces (or bulk), as there are no other charges. Let’s see if the core bulk faces to inside the cylinder!

\[ E = \frac{\rho}{\varepsilon_0} \text{, and } J = \frac{1}{r} \cdot \hat{r}, \text{ to ensure that } \]

\[ J \cdot 2\pi r \cdot L = \text{constant} (r) \text{ - that is, the same current flows through any the surface of any cylinder coaxial with the original charge conservation). This means that } E = \frac{\rho}{\varepsilon_0} \text{, and coincides with the field of uniform line charge at the axis of the cylinder. By uniqueness theorem, we know that this is the only charge distribution that can create such a field, but it has to be physically located on the outer surface of the cylinder. Finally, we conclude that } \]

\[ \vec{E} = \frac{1}{\varepsilon_0} \cdot \frac{\lambda}{r} \cdot \hat{r}, \lambda = \text{linear charge density}. \]

From here we get
\[ V = \phi_i - \phi_0 = -\int_a^b \vec{E} \cdot d\vec{r} = \int_a^b \frac{d\phi}{\varepsilon_0} \cdot \frac{\lambda}{2\pi r} = \frac{\lambda}{2\pi \varepsilon_0} \ln \frac{b}{a}, \]

and
\[ I = 2\pi r \cdot L \cdot J = 2\pi r \cdot L \cdot J \cdot \frac{\lambda}{2\pi \varepsilon_0 f}. \]
Finally, the resistance $R$ is given by

$$ R = \frac{\nu}{I} = \frac{1}{\sigma \varepsilon_0} \cdot \frac{\ln \frac{b}{a}}{1 - \sigma} \cdot \frac{1}{1/\varepsilon_0} = \frac{\rho \ln \frac{b}{a}}{2 \pi L} $$

Let's check the $b-a \ll a$ limit:

$$ \ln \frac{b}{a} \approx \ln \left(\frac{b-a}{a^2} + 1\right) \approx \frac{b-a}{a} \quad \text{(Taylor expansion)} $$

$$ R \approx \frac{\rho}{L} \frac{b-a}{a} \quad \text{as expected.} $$

**Flext heating**

It is well known that conductors heat up when a current is passing through them. How much heat is dissipated per unit time (dissipated power)?

If current $I$ flows through the conductor, every $dt$ of time there is a charge $\Delta Q = I dt$ passing through any cross section of the conductor. Effectively, this looks like charge $\Delta Q$ is taken from the left end to the right end of the sample. The work done by the electric field is

$$ \Delta W = \Delta Q \mathcal{E}_L - \Delta Q \mathcal{E}_R = \Delta Q \mathcal{E}_V \quad \text{. This work must be dissipated as heat, as the electron liquid} $$
is not accelerated on average. Thus the dissipate power is
\[ P = \frac{\Delta W}{\Delta t} = V \cdot \frac{\Delta Q}{\Delta t} = I \cdot V \]
- Joule heat

\[ [P] = \left[ \frac{\text{C}}{\text{t} \cdot \text{V}} \right] = \left[ \frac{\text{C} \cdot \text{V}}{\text{t}} \right] = \text{W (watts)} \]

If Ohm's law holds for the particular conductor, then \( IR = V \) and

\[ P = I^2 R = \frac{V^2}{R} \]

- which expression should be used in a particular situation is dictated by the experimental conditions: if \( I \) is kept constant, then \( P \propto R \), if we hold \( V \) constant then \( P \propto \frac{1}{R} \) (no current and dissipation as \( R \to \infty \)).

Differential form of Joule heating expression:

We can define local heat dissipated power per unit volume:

consider a small volume along the current flow. There we can apply \( P = I \cdot V \), where

\[ I = \int_{A} \mathbf{J} \cdot d\mathbf{A}, \quad V = E dL \quad \text{and} \quad P = \int_{V} \mathbf{E} \cdot d\mathbf{S} \text{. But} \]

\[ \mathbf{J} = \sigma \mathbf{E} \quad \text{so we get} \quad P = \sigma E^2 \cdot \frac{dS \cdot dL}{V} \quad \text{watts per unit volume element} \]
In other words, the dissipated power per unit volume is \[ P = \frac{1}{2} \oint E^2 \, dA \] if Ohm's law holds.

**Example:** discharge of a capacitor.

If we connect a capacitor to a resistor, such that there is a current path between the plates, how fast will the capacitor lose its charge?

Assuming connecting wires have negligible resistance, we note that the potential difference across the resistor coincides with the voltage on the capacitor. Then the current through the resistor is \( V = \frac{Q}{C} \)

\[ I = \frac{V}{R} \text{; } V = \frac{Q}{C} \Rightarrow I = \frac{Q}{RC} \text{, } Q \text{- charge on the "+" plate}. \]

At the same time, the current in the circuit gives the (negative) rate of change of the charge on the positive plate:

\[ I = -\frac{dQ}{dt} > 0 \text{ as } \frac{dQ}{dt} < 0. \]

Finally,

\[ \begin{cases} \dot{Q} = -\frac{Q}{RC} \\ Q(0) = Q_0 \end{cases} \Rightarrow Q = Q_0 e^{-t/RC} \]
We see that there is a typical discharge time scale \( t_{ee} = RC \), where the charge on the capacitor is
\[
[\epsilon_{ee}] = \frac{V}{A} \cdot \frac{C}{V} = \frac{C}{Ct} = t. \sqrt{V}.
\]

\[ I(t) = -\dot{Q} = \frac{Q_0}{RC} e^{-\frac{t}{RC}}. \]

The total dissipated energy:
\[
\frac{dW}{dt} = I \cdot V = I^2 \cdot R = \frac{Q_0^2}{RC} e^{-\frac{2t}{RC}}.
\]

\[ W = \int_0^\infty \frac{dW}{dt} = \frac{Q_0^2}{2C} \] — exactly the total energy that was stored inside the capacitor, thus should've been the answer by energy conservation.