Lecture 24 + 25 + 26

Magnetic field in vacuum

Just like we know that charges interact with each other (and we use this knowledge to build the field of electromagnetics), it is known that current-carrying wires also interact. To explain these and related facts, we need to assume the existence of another field that
- acts on moving charges (exerts force)
- is exerted by moving charges.

We call such a field "magnetic" and conjecture that it acts on a moving charge with a force

$$\vec{F} = q \vec{v} \times \vec{B},$$

where \( \vec{B} \) is the vector of the magnetic field.

When both \( \vec{E} \) and \( \vec{B} \) fields are present, the total force acting on a charge \( q \) is

$$\vec{F} = q \vec{E} + q \vec{v} \times \vec{B}.$$ - Lorentz force.

Unit: \([B] = \dfrac{N}{C \cdot M/s} = \dfrac{N}{m \cdot A} = \text{T (tesla)} \) (Earth's B-field is tens of mT. It is a relatively large field).
As was said, \( \mathbf{B} \)-field is both created by the moving charges, and exerts force on them. Let us first consider the latter aspect.

Cyclotron motion in a uniform \( \mathbf{B} \)-field. Consider a charge whose velocity is \( \mathbf{v} \) and \( q > 0 \) for definiteness.

Since \( \mathbf{v} \times \mathbf{B} \) is \( \mathbf{v} \), and the \( \mathbf{B} \) field does not change the magnitude of \( \mathbf{v} \), but only changes the direction. If \( \mathbf{B} \) is uniform, the particle would just go along a circle: what is the angular frequency?

\[
\omega = \frac{q \mathbf{v} \cdot \mathbf{B}}{m} \Rightarrow \\
\omega = \frac{q \mathbf{v} \cdot \mathbf{B}}{m} \Rightarrow \frac{v}{c} = \frac{qB}{m} \quad \text{v-independent}
\]

where \( \omega \) - cyclotron frequency.

The radius of the circle does depend on particle's momentum:

\[
R = \frac{m \mathbf{v}}{qB} = \frac{\mathbf{p}}{qB}
\]

When one considers the quantum case, these closed orbits lead to quantized energy levels (as mentioned, sort of), and are the ultimate reason for the quantum Hall effect.
The Hall effect
(E. Hall, 1879)

The HE represents the case where the current and B field are not parallel to each other. That is:

\[ \mathbf{J} = \sigma \mathbf{E}, \text{ and tensor } \sigma \text{ has off-diagonal components.} \]

Consider current flowing in a sample in the x-direction. If the charge of the carriers is \( q \), there is a Lorentz force acting on them due to B-field:

\[ \mathbf{F}_L = q \mathbf{v} \times \mathbf{B} = q \mathbf{v} \mathbf{B} \hat{\mathbf{e}}_x \times \hat{\mathbf{e}}_z = q \mathbf{v} \mathbf{B} \hat{\mathbf{e}}_y. \]

This force will deflect the charges in the \( \pm \hat{\mathbf{e}}_y \) direction (depending on the sign of \( q \)), and lead to charge accumulation on sample sides. This will continue until there is a strong enough \( E_y \) to counterbalance \( \mathbf{F}_L \):

\[ q \mathbf{E}_y + \mathbf{F}_L = 0 \Rightarrow q \mathbf{E}_y - q \mathbf{v} \mathbf{B} = 0 \Rightarrow \mathbf{E}_y = \mathbf{v} \mathbf{B}. \]

Thus:

\[ \mathbf{J}_x = q \mathbf{v} \mathbf{E}_y = q \mathbf{v} \mathbf{B} \hat{\mathbf{e}}_x \Rightarrow \mathbf{E}_y = \frac{B}{nq} \mathbf{J}_x. \]

This corresponds to an off-diagonal element of \( \mathbf{\sigma} \):

\[ \sigma_{yx} = \frac{E_y}{J_x} = \frac{B}{nq} \]

Hall resistivity.

NB: depends on \( q \), not \( q^2 \) ⇒ can measure sign of \( q \).
Overall conclusion: when passing current \( I \) in \( B \), one observes a transverse voltage \( \hat{E} \perp \hat{j}, \hat{B} \).

Let us now consider the reverse problem: we fix the electric field in \( y \)-direction, and see if it causes any current (in the \( y \)-direction).

Note: normally, impurities must be taken into account in such a problem. We assume that the sample is clean, that is \( \omega_c \gg \Omega \), where \( \omega_c \) is the cyclotron frequency. Then we will get null current only, which simplifies things.

**Equations of motion:**

\[
\begin{aligned}
m \ddot{\vec{r}} &= q \cdot \vec{E} + q \cdot \vec{v} \times \vec{B} \\
x(0) = y(0) = 0 \\
\vec{v} \times \vec{B} &= \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
0 & 0 & 0 \\
0 & 0 & B \\
\end{vmatrix} = B (\hat{v}_y \hat{e}_z - \hat{v}_z \hat{e}_y)
\end{aligned}
\]

Thus we get

\[
\begin{aligned}
m \ddot{x} &= q.B \cdot \ddot{v}_y = qB \dddot{y} \\
m \ddot{y} &= q.E - q.B \cdot \dddot{x} = qE - q.B \dddot{x} \\
\dot{x} &= \omega_c \cdot \dot{y} \\
\dot{y} &= \frac{q}{m} E - \omega_c x = \omega_c \left( \frac{E}{B} - \dot{x} \right)
\end{aligned}
\]

\[ \omega_c = \frac{qB}{m} \]
**Observation:** if \( \dot{x} = \frac{E}{B} \), we set \( \ddot{y} = 0, \dot{x} = 0 \) \( \Rightarrow \dot{y} = 0. \)

This is a solution of these equations, but it does not fit our initial conditions. Yet it shows that \( \frac{E}{B} \) is a special velocity on the problem.

To solve this system of differential equations, we write

\[ \ddot{x} = \omega_c \dot{y} = \omega_c^2 \left( \frac{E}{B} - \dot{x} \right) \]

Seek solution in the form of

\[ x(t) = \frac{E}{B} t + \ddot{x}(t) ; \]

\[ \ddot{x} = -\omega_c^2 \dot{x} \quad \text{or} \quad \frac{d}{dt} \left( \ddot{x} + \omega_c^2 \dot{x} \right) = 0. \]

Finally, we satisfy this equation by writing

\[ \ddot{x} = \text{(general solution of } \ddot{x} + \omega_c^2 \dot{x} = 0 \text{)} + \text{const.} \]

\[ x = A \cos \omega_c t + G \sin \omega_c t + C, \quad \text{or} \]

\[ x(t) = A \cos \omega_c t + G \sin \omega_c t + C + \frac{E}{B} t. \]  

Then

\[ \dot{y}(t) = \frac{1}{\omega_c} \cdot \dot{x} = -\omega_c A \cos \omega_c t - \omega_c G \sin \omega_c t \Rightarrow \]

\[ y(t) = G \cos \omega_c t - A \sin \omega_c t + D \]  

integration constant.
Now we have to fix all the constants:

\[ x(0) = 0 \Rightarrow A + C = 0. \]
\[ y(0) = 0 \Rightarrow G + D = 0. \]
\[ \dot{x}(0) = 0 \Rightarrow \omega AC + \frac{E}{B} = 0. \]
\[ \dot{y}(0) = 0 \Rightarrow A = 0. \]

Finally, let \( A = 0, C = 0 \),
\[ G = \omega_c \frac{E}{B}, D = \frac{1}{\omega_c} \frac{E}{B} \]
and we get
\[ x = \frac{E}{B} t - \frac{1}{\omega_c} \frac{E}{B} \sin \omega_c t = \frac{1}{\omega_c B} \left( \omega t - \sin \omega t \right). \]
\[ y = \frac{1}{\omega_c B} \left( \cos \omega_c t + 1 \right) = \frac{1}{\omega_c B} \left( 1 - \cos \omega t \right) \]

These equations define the trajectory of the particle.

We can see that the motion in y-direction is bounded between \( y = 0 \) and \( y = \frac{1}{\omega_c} \frac{E}{B} \), but the motion along \( x \) is unbounded.

To get an idea of the trajectory, consider
\[ \left( x - \frac{E}{B} t \right)^2 + \left( y - \frac{1}{\omega_c} \frac{E}{B} \right)^2 = \left( \frac{1}{\omega_c B} \right)^2 \]

We see that it's an equation of a circle with radius \( R = \frac{1}{\omega_c} \frac{E}{B} \), whose center has \( y = R \), and moves to the right with a speed of \( \frac{E}{B} \).

\[ 2k \uparrow \quad \text{cycloid} \]
Clearly, there is an average drift in the \( x \)-direction, with a speed of \( \bar{v}_x = \frac{E}{B} \).

This translates into \( \bar{j}_x = \frac{E}{B} \bar{v}_x = \frac{E \rho}{B} \). \( E_y \), or we are back to \( E_y = \frac{B}{\rho} \bar{j}_x \).
The Biot–Savart law

We start our discussion of how currents produce magnetic fields with the case of a thin wire carrying a steady current (which means we are dealing with a steady magnetic field, even though the charges are moving).

The field of a wire is given by

\[ \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \mathbf{dl}' \times \frac{\mathbf{I}}{r^2}, \]

where \( \mathbf{I} = \mathbf{I} - \mathbf{I}' \), \( \mathbf{r} = \mathbf{r} - \mathbf{r}' \), \( \mathbf{I} \) being the observation point, \( \mathbf{r}' \) - the remaining position of a segment of a wire (see the figure). \( \mathbf{I} = I \mathbf{dl}' \), directed current at point \( \mathbf{r}' \).

This formula has the name of the Biot–Savart law. Using \( \mathbf{I} = I \mathbf{dl}' \), we can write

\[ \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \int \frac{\mathbf{dl}' \times \mathbf{I}}{r^2} \] - another form of B.

\( \mu_0 = 4\pi \cdot 10^{-7} \frac{N}{A} \) - magnetic permeability of vacuum,

(sort of an analogy to in the context of magnetic vortices).
Application: field of an infinite straight wire

The B-S law tells us that the field lines of \( \vec{B} \) are circles centered at the wire (see fig), that is, at point 1 the \( \vec{B} \)-field points out of sheet. Its magnitude is

\[
\vec{d}l = dx \cdot \hat{y}
\]

\[
\begin{align*}
\vec{B}(d) &= \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} dx \cdot \frac{3mI}{x^2+d^2} = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} dx \frac{1}{(x^2+d^2)^{3/2}} \\
&= \frac{\mu_0 I}{4\pi d} \int_{-\infty}^{\infty} \frac{1}{(x^2)^{3/2}} = \frac{\mu_0 I}{2\pi d}
\end{align*}
\]

The \( d \)-dependence is similar to that of the \( \vec{E} \)-field at a thin charged line, but the field lines look very different: they are closed - do not have a beginning or end, and they run such sheet there is net circulation of \( \vec{B} \) around the wire.

We start to suspect that \( \vec{D} \cdot \vec{B} = 0 \) (closed field lines), but \( \vec{E} \times \vec{B} \neq 0 \) (there is some circulation).
**Force acting on a current carrying wire**

If we have a wire with current placed in a magnetic field, there will be a force acting on it, due to the magnetic Lorentz force acting on many charge carriers:

Force acting on a segment of the wire is

\[ \mathbf{F} = \sum q \, \mathbf{v} \times \mathbf{B} = q \, \mathbf{v} \times \mathbf{B} \cdot \mathbf{n} \, \# \, d\ell; \]

\( \# \) numbers charge carriers within the segment.

But \( nq \, \mathbf{v} = \mathbf{j} \) is the current density, and \( \mathbf{j} \, d\ell = I \) is the current in the segment (with its direction specified).

Thus \( \mathbf{F} = \mathbf{I} \times \mathbf{B} \, d\ell \) or we can introduce the force per unit length:

\[ f = \frac{d\mathbf{F}}{d\ell} = \mathbf{I} \times \mathbf{B}; \]

Total force:

\[ \mathbf{F} = \int d\ell \cdot \mathbf{I} \times \mathbf{B} = \mathbf{I} \int d\ell \times \mathbf{B}. \]
Application:

Let us calculate the force (per unit length) between two parallel wires.

The wire 1 creates $\vec{B}$-field that points into the sheet at the location of wire 2.

$\vec{I}_2 \times \vec{B}$ points toward the 1st wire for $\vec{I}_2 \parallel \vec{I}_1$, and away from it for $\vec{I}_1 \perp \vec{I}_2$.

Thus anti-parallel currents repel, and parallel current attract.

$$f = \frac{I_2}{2\pi d} \cdot \frac{\mu_0}{2\pi d} \cdot I_1 = \frac{\mu_0}{2\pi d} I_1 I_2$$

The overall sign is determined by the relative signs of $I_1, I_2$ - as explained above: $I_1 I_2 > 0$ means attraction, $I_1 I_2 < 0$ - repulsion.