Magnetic dipole in a magnetic field.

a) Torque acting on a dipole:

For the sake of the argument, let us consider a rectangular loop ax-b with current \( i \) flowing in it. A more complicated loop can be split into small rectangular loops, and the resultant torque will be just the total torque acting on all loops if we can neglect variation of \( B \)-field across this large "composite" loop.

For definiteness orient \( B \) along the \( z \)-axis and assume that the loop was initially in the \( x-y \) plane but is rotated around the \( y \)-axis by angle \( \theta \):

The forces that act on sides (1) and (3) try to "stretch" the loop, and do not create any torque.

There is a pair of forces that act on edges (2) and (4) that do create a torque directed along (-z) direction (they try to orient the loop along \( z \)).

\[
F_2 = F_4 = I \cdot B \cdot l \cdot \hat{e}_z \quad \tau = (F_2 + F_4) \cdot \frac{d}{2} \cdot \sin \theta = I \cdot B \cdot l \cdot \hat{e}_z \cdot \sin \theta
\]
But \( I \vec{a} \cdot \vec{B} = m \) - magnetic moment of the loop, and we get \( \vec{L} = I \vec{m} \times \vec{B} \), oriented to both \( \vec{B} \) and \( m \). It is easy to show that this is just a particular case of the following expression:

\[
\vec{L} = \vec{m} \times \vec{B}
\]

Just like in the electric dipole case, the field tries to orient the dipole parallel to itself (when \( \vec{m} \perp \vec{B} \), the torque also vanishes but the orientation is unstable).

b) Force acting on a dipole

For a closed loop, \( \vec{F} = \int \vec{J} \times \vec{E} \) is zero. \( \int \vec{B} \times \vec{I} \) is uniform. Thus a force acts on a magnetic dipole only in a non-uniform \( B \)-field.

For simplicity, consider again a planar rectangular loop of infinite small extent, located in the \( x-y \) plane as shown in the figure:

Let us consider forces acting on pairs of opposite sides:

\[
\vec{F}_a + \vec{F}_b = I \int \vec{x} \times \vec{B}(x', y, z) - I \int \vec{x} \times \vec{B}(x', y + \Delta y, z)
\]

\[
\int \vec{x} \times \vec{B}(x', y, z) = \frac{B(x, y, z)}{x' - x} \quad (\text{same } B \text{-field})
\]

\[
\int \vec{x} \times \vec{B}(x', y, z) = \frac{B(x, y, z)}{y - (y + \Delta y)}
\]
\[ \mathbf{F}_{\text{tot}} = \mathbf{\nabla} \left( m_2 \mathbf{B}_z \right) + \mathbf{\hat{y}} \cdot \frac{\partial}{\partial y} \left( m_2 \mathbf{B}_z \right) - \mathbf{\hat{z}} \left( \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} \right) m_2 \]

But we know that \( \mathbf{\nabla} \cdot \mathbf{B} = 0 \), thus \( \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \), and we finally get

\[ \mathbf{F}_{\text{tot}} = \left( \mathbf{\hat{x}} \frac{\partial}{\partial x} + \mathbf{\hat{y}} \frac{\partial}{\partial y} + \mathbf{\hat{z}} \frac{\partial}{\partial z} \right) m_2 \mathbf{B}_z = \mathbf{\nabla} \left( m_2 \mathbf{B}_z \right) \]

But the direction of \( \mathbf{m} \) was chosen to be the \( x \)-axis. If we chose it to be, say \( \mathbf{\hat{y}} \), we would get

\[ \mathbf{F}_{\text{tot}} = \mathbf{\nabla} \left( m_2 \mathbf{B}_x \right) \]
It may look like this is different in form from the electric case:
\[ \mathbf{F}_e = (\mathbf{p} \times \mathbf{E}). \]

However, since \( \mathbf{B} \times \mathbf{E} = 0 \), we obtain that
\[ \mathbf{p} \times (\mathbf{B} \times \mathbf{E}) = 0 = \mathbf{B} (\mathbf{p} \times \mathbf{E}) - (\mathbf{p} \times \mathbf{E}) \mathbf{E} \Rightarrow (\mathbf{p} \times \mathbf{E}) = \mathbf{B} (\mathbf{p} \times \mathbf{E}), \]
\[ \mathbf{F}_e = \mathbf{B} (\mathbf{p} \times \mathbf{E}) - (\mathbf{p} \times \mathbf{E}) \mathbf{E}. \]

c) Energy of dipoles in external fields:

The fact that forces acting on electric and magnetic dipoles, in \( E \) and \( B \)- fields, respectively, are given by gradients of some functions, suggests that these forces are potential. They can be assigned a potential energy function such that
\[ \mathbf{F} = -\nabla U \text{ where } U \text{ for the magnetic and electric cases is} \]
\[
\begin{align*}
U_B &= -\mathbf{p} \cdot \mathbf{B} \\
U_E &= -\mathbf{p} \cdot \mathbf{E}
\end{align*}
\]
These expressions tell us that for rigid dipoles the orientation along the field has the least energy, and is thus stable.
2) Orbital response to magnetic field

As we have just seen, a magnetic field tries to orient dipoles along itself. This is the so-called "papa magnet." tendency (see below for the explanation of the term).

But it also can change the magnetic moment itself and the change is usually opposite to the direction of \( \mathbf{B} \). Let us see how this happens using a simple model.

Consider an electron rotating around an ion as shown in the figure. Since the electron is negatively charged, the magnetic moment is directed downwards.

When the \( B \)-field is turned on in the \( z \)-direction (i.e., the trajectory), there is an additional force towards the ion (the Lorentz force), which means the electron goes . It can be proven that upon switching on the \( B \)-field, the radius of the trajectory does not change. Then, from the Newton's second law we have:

\[
\frac{m \mathbf{v}^2}{R} = F_0 + \mathbf{e} \mathbf{v} \cdot \mathbf{B}
\]

\[
\frac{m \mathbf{v}^2}{R} = F_0 + 1/e \mathbf{v} \cdot \mathbf{B} \quad \text{after, } v > v_0.
\]
Subtracting the first equation from the second we get:

\[
\frac{m}{R} (v^2 - u^2) = le\mathbf{v} \mathbf{B},
\]

assuming \(v = v_0 \ll u_0\) (near field) we write

\[
\frac{m}{R} (v - v_0)(v + v_0) < \frac{2m v_0 (v - v_0)}{R} \Rightarrow v - v_0 = \frac{le |B|}{2mn} R > 0.
\]

Let \(v - v_0 \approx le |B| B\)

Since \(v > v_0\), the maximum moment is larger in magnitude but the charge points against \(B\):

\[
\mathbf{M}_f - \mathbf{M}_i = \left( \frac{le}{2mR} \mathbf{v}_f - \frac{le}{2mR} \mathbf{v}_i \right) \mathbf{z} = -\frac{le |B|^2}{2mR} \frac{le |B|}{2mn} \mathbf{z} = \left( \frac{\mathbf{e}}{\mathbf{x}} \right) = \frac{\mathbf{e} \mathbf{r}}{\mathbf{r}} - \text{current in the "loop"}, \mathbf{z} = -\frac{e^2 R^2}{2 \pi m} \mathbf{B} \quad (\mathbf{z} = \mathbf{B}).
\]

Now if there is a bunch of such loops oriented randomly, the initial magnetic moment of the sample is zero, but each loop gets a change in the magnetic moment that is opposite to \(B\) and thus there is a net "diamagnetic" response: the induced net magnetic moment is \(\mathbf{I} \mathbf{B}\).

Why don't loops orient themselves along \(B\) in the first place? Well, they do. The question is what tendency prevails:

\[
\text{Why don't loops orient themselves along } B \text{ in the first place? Well, they do. The question is what tendency prevails:}
\]
Paramagnetism aligns with $\mathbf{B}$, or "diamagnetism" generation of $\mathbf{H}$ opposite to $\mathbf{B}$. In different substances the winner is different.

**Magnetization**

$$\mathbf{M} = \text{magnetic dipole moment per unit volume}$$

This is a very similar definition compared to the one we had in the case of electric polarizability. Specifically:

$$\mathbf{M} = \frac{\sum \mathbf{m}_i}{\Delta V}$$

- Delensity of dipole moments.

As far as the response to magnetic field is concerned,

- $\mathbf{M} \propto \mathbf{B}$ in paramagnetic materials
  - $(\mathbf{M} \propto -\mathbf{B}^2$ - they like large $\mathbf{B}$-fields)
  - Low energy minima.

- $\mathbf{M} \propto -\mathbf{B}$ in diamagnetic materials
  - $(\mathbf{M} \propto +\mathbf{B}^2$ - they try to avoid regions with large $\mathbf{B}$-fields).

$\mathbf{M}$ exists even without $\mathbf{B}$ on ferromagnets. These are said to have spontaneous magnetization.