Electrodynamics

Summary of electrostatics and magnetostatics in vacuum: Maxwell equations for static fields.
\[ \nabla \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho \quad \nabla \cdot \vec{B} = 0 \]
\[ \nabla \times \vec{E} = 0 \quad \nabla \times \vec{B} = \mu_0 \vec{J} \]

It turns out that the equations for the divergence of \( \vec{E} \) and \( \vec{B} \) remain unchanged in the dynamic case, while the ones for curls do change: it turns out that time-varying electric fields can cause magnetic fields, and vice versa.

Getting ahead of ourselves, let's put down the four Maxwell equations in vacuum, but for time-varying fields:

\( \text{Gauss's law) \quad \nabla \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho } \)
\( \text{Nameless law) \quad \nabla \cdot \vec{B} = 0 } \)
\( \text{(Faraday's law) \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} } \)
\( \text{(Ampere's law) \quad \nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} } \)

\[ \text{written by Heaviside, actually} \]

We will start with modifications of Ampere's law.

It is easy to see why the static version of it cannot hold universally.

Let's apply \( \nabla \times \) to both sides of the equations:
\[ \nabla \cdot ( \nabla \times \mathbf{B} ) = 0. \]

\[ \nabla \cdot \mathbf{D} \neq 0 \text{ for time-varying currents in general!} \]

To see an example, consider a discharging capacitor:

\[ \begin{array}{c}
-\mathbf{Q} \\
\mathbf{i} = 0 \text{ near the plates;} \\
\text{this is a general statement:} \\
\text{time-varying charges lead}
\end{array} \]

\[ \begin{array}{c}
to current densities with \\
non-zero divergence. Let's derive the corresponding \\
\underline{continuity equation}: \\
\text{Total charge inside an arbitrary} \\
\text{volume, } V:\n\]

\[ Q = \int_V dV \cdot \mathbf{p} \]

If \( Q \) changes with time, it implies that either charges flow into the volume, or out of it; they cannot simply appear or disappear due to \underline{charge conservation}. Thus, to calculate \( \frac{dQ}{dt} \), we simply need to count how much charge crosses the boundary of the volume per unit time:

\[ \text{if } \mathbf{j} > 0, \text{ the charges are leaving the volume, thus} \]

\[ \frac{dQ}{dt} = \int_{\partial V} \mathbf{j} \cdot d\mathbf{s} = -\oint_{\partial V} \mathbf{B} \cdot d\mathbf{s} \]

\[ \text{total current flowing out of the volume} \]
thus we get that
\[ \int \text{div} \cdot \frac{\partial \mathbf{E}}{\partial t} = -\int \text{curl} \cdot \mathbf{j} \] 
By Gauss's theorem, \( \int \text{curl} \cdot \mathbf{j} = \int \text{div} \cdot \mathbf{A} \cdot \mathbf{j} \), thus
\[ \int \text{div} \cdot \frac{\partial \mathbf{E}}{\partial t} = -\int \text{curl} \cdot \mathbf{j} \Rightarrow \frac{\partial \mathbf{E}}{\partial t} - \nabla \cdot \mathbf{j} = 0 \]
This is the celebrated \textit{continuity equation}. It holds for any conserved quantity, not just charge density.

Now it is clear how to fix the equation for \( \mathbf{E} \):
\[ \nabla \cdot \mathbf{j} = -\frac{\partial \mathbf{E}}{\partial t} = -\varepsilon_0 \nabla \cdot \mathbf{E} \]
Thus if we add \( \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \) to the r.h.s. of the equation, we will have vanishing divergences on both sides:
\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \]
Check: \( \nabla (\varepsilon_0 \mathbf{E}) \propto \nabla (\mu_0 \mathbf{j} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}) = \mu_0 (\nabla \cdot \mathbf{j} + \frac{\partial \mathbf{E}}{\partial t}) = 0 \). \( \checkmark \)

\( \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \) is called the \textit{displacement current}.

We see that \( \nabla \cdot (\mathbf{j} + \text{disp. current}) = 0 \), that is, the field lines of this quantity are \textit{closed loops}. 