Electromagnetic waves in vacuum

Wave equation:

We will consider solutions to Maxwell equations in the absence of matter (charges or currents):

\[ \nabla \cdot \mathbf{E} = 0 \quad \nabla \cdot \mathbf{B} = 0 \]

\[ \nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} \quad \nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \]

Let us reduce these equations to those for \( \mathbf{E} \) and \( \mathbf{B} \) separately:

\[ \nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t} (\nabla \times \mathbf{B}) = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \]

On the other hand,

\[ \nabla \times (\nabla \times \mathbf{E}) = \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = - \Delta \mathbf{E} \]

Thus, no charges

we get

\[ \Delta \mathbf{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \]

Analogously we can obtain

\[ \Delta \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} \]

The equation of this form \( \frac{\partial^2 \mathbf{E}}{\partial x^2} = \frac{\partial^2 \mathbf{E}}{\partial t^2} \) is known as the wave equation in the literature. It describes propagation of disturbances in space and time with a speed of \( c \).
The general solution to this equation has the following form:

\[ f(x) = g_1(x + ct) + g_2(x - ct). \]

Indeed, \( \frac{\partial^2 f}{\partial x^2} = g''_1(x + ct) + g''_2(x - ct) \) and \( \frac{\partial^2 f}{\partial t^2} = c^2 g''_1(x + ct) + c^2 g''_2(x - ct) \).

Among all possible solutions, a special role is played by monochromatic solutions:

\[ f_c = f_0 \cos(kx + \omega t + \delta) \]

where \( k \) is the wave number of the wave, \( \omega \) is its angular frequency, and \( \delta \) is a phase shift of the wave.

In order for this wave to satisfy the wave equation, we must have \( \omega = kc \). Such a relation between a wave's frequency and its wave number is called the dispersion relation. In the present case, the dispersion is linear.

The wave number and angular frequency of a wave are related to the wave length and period of the wave.
Let's sketch \( f(x,t) \):

\[
\begin{align*}
 t = 0: \quad f(x,0) &= \cos(kx + \phi) \\
 t = \frac{2\pi}{\omega}: \quad f(x, \frac{2\pi}{\omega}) &= \cos(kx - \omega \cdot \frac{2\pi}{\omega} + \phi) = f(x,0) \Rightarrow \frac{2\pi}{\omega} &= \text{period of the wave.}
\end{align*}
\]

**Complex notation:**

\[
f(x,t) = \text{Re} \cdot f_0 \cdot e^{i(kx-\omega t)}
\]

\[
= \text{Re} \cdot f_0 \cdot \cos(kx - \omega t + \phi) \rightarrow \text{back to the original form.}
\]

This form is very convenient when one manipulates signals (addition, etc.).

Thus we might write \( f \cdot e^{-i\omega t} \), but imply that the physical signal is the real part of the complex one.

**Plane waves:**

For the 3D wave equation, \( \Delta f = \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2} \), the plane wave looks like

\[
f = f_0 \cdot e^{i(kz-\omega t)}, \quad \vec{k}(k_x, k_y, k_z) - \text{wave vector}
\]

\( \vec{k} \) determines the direction of wave propagation.
Back to the E-M fields:

\[ \Delta \mathbf{E} = \mu_0 \mathbf{E} \frac{\partial \mathbf{E}}{\partial t} \]

\[ \Delta \mathbf{B} = \mu_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} \]

We see that there is a speed hidden in Maxwell equations: \( C = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \) \( \approx 2.99 \times 10^8 \text{ m/s} \) - the speed of light in vacuum. This turns out to be the maximal speed in nature, which can correspond to a physical signal transmission (one can cook up higher speeds, but one won't be able to transmit information at those speeds).

**Plane wave solutions:**

\[ \mathbf{E} = \mathbf{E}_0 \, e^{i(k \mathbf{r} - \omega t)} \]

\[ \mathbf{B} = \mathbf{B}_0 \, e^{i(k \mathbf{r} - \omega t)} \]

Are \( \mathbf{E}_0 \) and \( \mathbf{B}_0 \) arbitrary? No.

\[ \mathbf{E}_0 = E_0 \mathbf{n} \], \( \mathbf{n}(|\mathbf{n}|=1) \) - direction of \( E \)-oscillations, called "polarization".

\( E(\mathbf{n}t) \), however, must satisfy \( \nabla \cdot \mathbf{E} = 0 \), thus:

\[ \nabla \mathbf{E} = (E_0 \mathbf{n}) \cdot e^{i(k \mathbf{r} - \omega t)} = iE_0 (\mathbf{r} \cdot \mathbf{n}) \, e^{i(k \mathbf{r} - \omega t)} \]

Therefore, \( \mathbf{n} \cdot \mathbf{k} = 0 \): \( \mathbf{n} \) must be \( \perp \) to \( \mathbf{k} \), that is, \( \perp \) to the propagation direction. E-M waves in vacuum are transverse.
Further, $\mathbf{B}_0$ is related to $\mathbf{E}_0$ due to the Faraday's law:
\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \Rightarrow \text{i}w \mathbf{E}_0 = \text{i}k \mathbf{E}_0 \times \mathbf{n}, \text{or} \quad \mathbf{E}_0 = \frac{1}{\text{i}k} (\mathbf{E}_0 \times \mathbf{n}) \mathbf{n} = \frac{1}{\text{i}c} \mathbf{E}_0 \times \mathbf{n}.
\]

That is, $\mathbf{B}$ is perpendicular to both $\mathbf{x}$ and $\mathbf{n}$, the magnitude of $\mathbf{B}$ is $\frac{1}{c}$ (magnitude of $\mathbf{E}$).

Finally,
\[
\mathbf{E} = \mathbf{E}_0 \mathbf{n} \cdot e^{(\mathbf{kr}-\text{i}w)t}, \quad \text{where} \quad \mathbf{r} = \mathbf{r}.
\]
\[
\mathbf{B} = \frac{1}{\text{i}c} \mathbf{E}_0 (\mathbf{r} \times \mathbf{n}) \cdot e^{(\mathbf{kr}-\text{i}w)t}.
\]

Example: plane wave propagating in $x$ direction, polarized in $y$ direction:
\[
\mathbf{E} = \mathbf{E}_0 \mathbf{y} \cdot e^{(\mathbf{kr}-\text{i}w)t}
\]
\[
\mathbf{B} = \frac{\mathbf{E}_0}{\text{i}c} (\mathbf{r} \times \mathbf{y}) e^{(\mathbf{kr}-\text{i}w)t}.
\]