Electric charge and Electric Field.

a) Electric charge - a fundamental physical property of matter (or elementary particles) that allows it to create electromagnetic fields and experience a force when subject to them.

More operationally, the presence of an electric charge results in interaction of a charged body with other charged bodies.

Properties: (experimental observations)

- There are two types of electric charges, positive and negative (the specific assignment is only a convention). Charges of the same sign repel, the opposite attract each other.

- All charges are quantized in units of the elementary charge, $e > 0$, $e = 1.6 \times 10^{-19}$ Coulomb

Electron: $q_e = -e$
Proton: $q_p = e$
Neutron: $q_n = 0$.
Since the elementary charge is very small and usually (!) one deals with assemblies of many charges (macroscopic bodies), the quantization is often not important.

- the charge is conserved: it is impossible to change the total charge of an electrically insulated body (meaning no electric current flows through its boundary).

Compare to, say, the number of radioactive nuclei in a sample, that decays on time.

b) Coulomb's law

The point charge: a charged body whose dimensions are small compared to any other scale in the considered situation.

Coulomb experimentally discovered that the force between two point-like charged objects is proportional to the product of their charges (taking into account their signs), and inversely proportional to the square of the distance between them:

\[ F_{12} = \frac{k q_1 q_2}{r_{12}^2} \]
\[ F_1 = -C \frac{q_1 q_2}{r_{12}^2} \cdot \frac{\vec{r}_{12}}{r_{12}} = C \frac{q_1 q_2}{r_{21}^2} \cdot \frac{\vec{r}_{21}}{r_{21}} \]

- magnitude and sign
- direction.

\[ F_2 = C \frac{q_1 q_2}{r_{12}^2} \cdot \frac{\vec{r}_{12}}{r_{12}} \]

- If \( q_1 q_2 < 0 \) - attraction
- If \( q_1 q_2 > 0 \) - repulsion.

The proportionality coefficient, \( C \), depends on the system of units. In SI units, \( C = \frac{1}{4\pi \varepsilon_0} \), where

\[ \varepsilon_0 \approx 8.85 \cdot 10^{-12} \frac{F}{m \cdot m^2} \]

\( c) \) **Electric Field**

Defining an electric field, \( \vec{E}(r) \) at point \( \vec{r} \): consider a system of charges \( q_1 \ldots q_n \), and a test charge \( Q \).

\[ \vec{E}(\vec{r}) = \sum_{i=1}^{n} \frac{q_i}{4\pi \varepsilon_0} \frac{Q \cdot q_i}{|\vec{r} - \vec{r}_i|^2} \cdot \frac{\vec{r} - \vec{r}_i}{|\vec{r} - \vec{r}_i|} \]
1. \( \vec{E} \cdot \nabla \frac{1}{r} = \frac{9 \epsilon_0}{4 \pi} \left( \frac{\vec{r} - \vec{r}_0}{|\vec{r} - \vec{r}_0|^2} \right) \frac{\vec{r} - \vec{r}_0}{|\vec{r} - \vec{r}_0|} \)

Independent of \( \vec{r}_0 \), property of the point \( \vec{r} \) itself.

Suggest to define a new quantity, \( \vec{E}(\vec{r}) \):

\[
\vec{E}(\vec{r}) = \frac{\vec{F}(\vec{r})}{Q}.
\]

Electric field at point \( \vec{r} \) is defined as the ratio of an electric force acting on a test charge to the value of the charge.

This definition is very operational. Suggests that we know how to play the game of electromagnetism, but we do not really know why the players are there.

2) Electric field of a point charge

Let's place a point charge at the origin, and a test charge at \( \vec{r} \):

\[
\vec{F}_Q = \frac{1}{4 \pi \epsilon_0} \frac{9 \epsilon_0}{r^2} \vec{r} = \frac{1}{4 \pi \epsilon_0} \frac{9 \epsilon_0}{r^2} \vec{r}, \text{ thus}
\]
\[ \vec{E}(\vec{r}) = \frac{F_Q}{Q} = \frac{1}{4\pi \varepsilon_0} \frac{Q}{r^2} \cdot \hat{r} \]

\[ [E] = \frac{V}{m} \left( \text{volt/meter} \right) \]

e) **Superposition. Electric field of a dipole**

Superposition principle: For a system of charges, the electric field they create at point \( \vec{r} \) is the sum of the fields each of the charges creates in the absence of others:

\[ \vec{E}(\vec{r}) = \sum \vec{E}_Q(\vec{r}) \]

**Field of a dipole:** Consider two opposite charges, \( \pm q \), located at distance \( l \) away from each other:

Let us find a few representative values of its electric field (points A, B, C).
Clearly the field is symmetric around the axis of the dipole, so we can consider it only in the sheet's plane.

\[ E_A : \text{points to the right} \left( |E_{+q}| > |E_{-q}| \right) \]

\[
E_A = \frac{1}{4\pi \varepsilon_0} \frac{q}{(r - \frac{b}{2})^2} - \frac{1}{4\pi \varepsilon_0} \frac{q}{(r + \frac{b}{2})^2} =
\]

\[ = \frac{q}{4\pi \varepsilon_0} \frac{2rb}{(r^2 + (\frac{b}{2})^2)^2} \approx \frac{1}{4\pi \varepsilon_0} \frac{2q b}{r^3} \]

The combination \( \mathbf{p} = q\mathbf{\ell} \) is called the dipole moment of the dipole. It is a vector:

\[ \mathbf{p} = q\mathbf{\ell}, \text{ pointing from the negative charge toward the positive one. Then } \mathbf{E}_A = \frac{1}{4\pi \varepsilon_0} \frac{2\mathbf{p}}{r^3} \]

\[ E_B : \text{points to the right} \text{ (again, as at A) because} \]

\[ |E_{-q}| > |E_{+q}| \text{ and points to the right.} \]

\[ |E_B| = |E_A| \]

\[ \mathbf{E}_c : \text{the direction is sketched on the figure.} \]

\[
|\mathbf{E}_c| = 2 \cdot |E_{+q}| \cdot \sin \theta =
\]

\[ = 2 \cdot \frac{q}{4\pi \varepsilon_0} \frac{l/2}{\sqrt{r^2 + (l/2)^2}} \]
\[
\frac{1}{4\pi \varepsilon_0} \cdot \frac{q r}{r^3} = \frac{1}{r^2} \cdot \frac{F^2}{4\pi \varepsilon_0} = \frac{1}{r^2} E_0 = \frac{1}{r^2} E = \frac{1}{r^2} E_B.
\]

In principle, we can find an expression for the field at any point. We will get to this later.

f) **Continuous charge distribution.**

Often we will deal with large charge assemblies, not caring about precise positions of all charges, but only about charge density. This point of view is applicable when we are interested in fields at distances large compared to distances between a charges.

Example: 3D charge density:

\[
p = \frac{\sum_{i=1}^{N} q_i}{\Delta V} = q \cdot \frac{N}{\Delta V}
\]

\(\Delta V\) is physically small: dimensions smaller than any other in the problem but still containing a large number of particles.

Why is it useful? A macroscopic body can be split into volumes like that and sum up their fields taking a volume integral.
\[ \vec{E}(r) = \sum \frac{1}{4\pi\varepsilon_0} \frac{q_i}{|r - r_i|^2} \]

\[ \sum q_i = \rho \Delta V, \quad \Rightarrow \sum q_i = \rho \Delta V \]

Field of a point charge with \( q = \rho \Delta V \)

\[ = \frac{1}{4\pi\varepsilon_0} \frac{\rho \Delta V}{r^2} \]

\[ = \frac{1}{4\pi\varepsilon_0} \frac{\rho}{r^2} \]

Field over all \( \Delta V \)s get the total field.

**NB:** the dimensions of the macroscopic body itself need not be small compared to the distance to the observation point.

Let us calculate a few fields using these ideas.

**1D example:** charged infinite line with a constant charge density, \( \lambda \).

By symmetry, the field depends only on the radial distance to the line, and points radially.
we need only $\Delta E_\perp$, as $\Delta E_\parallel$ is cancelled when summed over all $\Delta x$.

$$\Delta E_\perp = \Delta E \cos \theta = \frac{1}{4\pi \varepsilon_0} \cdot \frac{\lambda \Delta x}{r^2 + x^2} \cdot \frac{r}{\sqrt{r^2 + x^2}} =$$

$$= \frac{\lambda}{4\pi \varepsilon_0} \cdot \frac{r}{\sqrt{r^2 + x^2}} \cdot \Delta x$$

$$E_{\text{total}} = \sum \Delta E_\perp = \int_0^\infty \Delta x \cdot \frac{\lambda}{4\pi \varepsilon_0} \cdot \frac{1}{\sqrt{r^2 + x^2}} =$$

$$= \frac{\lambda}{4\pi \varepsilon_0} \cdot \frac{1}{r} \int_0^\infty \frac{1}{\sqrt{r^2 + x^2}} x dx =$$

$$= \frac{1}{2\pi \varepsilon_0} \cdot \frac{\lambda}{r}$$

To get a better feeling of this result, consider the field of a finite line segment, calculated along the line dissecting it into two equal pieces.

$$E_{\text{total}} = \int_{-a/2}^{a/2} \frac{\lambda}{4\pi \varepsilon_0} \cdot \frac{r}{\sqrt{r^2 + x^2}} dx$$
Switching again to dimensionless variables, we get

$$E_{\text{total}} = \frac{1}{4\pi \varepsilon_0 r} \cdot \frac{1}{2r} \cdot \frac{1}{(1 + \frac{a^2}{r^2})^{3/2}}.$$

The limits of integration are now finite. We still can do the integral exactly, but let us consider a couple of limiting cases instead.

\(a \gg r\): in this case we are "very close" to the line and we should get back the expr. for an infinite line (we "cannot see the ends"). Indeed, for \(a \gg r\), \(\sqrt{r^2 + a^2} \gg 1\), and

$$\frac{1}{(1 + \frac{a^2}{r^2})^{3/2}} \approx 1,$$

so

$$E_{\text{total}} \approx \frac{1}{4\pi \varepsilon_0 r} \cdot \frac{1}{2r} \cdot \frac{1}{(1 + \frac{a^2}{r^2})^{3/2}} \approx \frac{1}{4\pi \varepsilon_0 r} \cdot \frac{1}{2r} \cdot \frac{1}{(1 + \frac{a^2}{r^2})^{3/2}}.$$

\(a \ll r\): in this case we have at the segment from far away and it looks like a point charge. Formally, this is obtained

$$\frac{1}{(1 + \frac{a^2}{r^2})^{3/2}} \approx \frac{1}{1 + 1} \approx 1,$$

$$E_{\text{total}} \approx \frac{1}{4\pi \varepsilon_0 r} \cdot \frac{1}{2r} \cdot \frac{1}{(1 + \frac{a^2}{r^2})^{3/2}} \approx \frac{1}{4\pi \varepsilon_0 r} \cdot \frac{1}{2r} \cdot \frac{1}{(1 + \frac{a^2}{r^2})^{3/2}}.$$

\(E_{\text{total}} = \frac{1}{4\pi \varepsilon_0} \cdot \frac{1}{r^2} \cdot (\alpha A) - \text{field of a point charge} \quad (\alpha A \text{ is the total charge of the object})\)
2D example: charged infinite plane with a uniform surface charge density $\sigma$.

By symmetry, the field points to the plane, away from it for $\sigma > 0$, and toward it for $\sigma < 0$.

To calculate the field in the simplest manner, we split the full plane into concentric circles with their centers located at the projection of the point in question onto the plane.

First, we calculate the field at $A$ from one of the "rings": we have to sum up vertical projections of fields from all pieces of the ring, as we did for a line segment (after all, $A$ is a curved line segment).

The charge of each piece is $\Delta q = \sigma \Delta \text{Area}$.

$\Delta \text{Area} = \sigma r \Delta \theta \Delta r$

area element in polar coordinates, see the Fig.

Let us obtain this expression for completeness:

Total area = $\pi r^2 - \pi (r-\Delta r)^2 = 2\pi r \Delta r - \pi \Delta r^2$
Thus if total area is \(2\pi r \, dr\), the area of a piece with angular dimension of \(d\phi\) is \(dA = d\phi \cdot r \, dr\),

\[
\sum dA = r \, dr \sum \frac{d\phi}{\Delta \phi} = 2\pi r \, dr
\]

Finally,

\[
\Delta E = \frac{1}{4\pi \varepsilon_0} \cdot \frac{1}{r^2 + d^2} \cdot \delta \, r \, \delta \, \phi \, \Delta \phi \quad (\text{see the fog for direction})
\]

(\(r\) now is the running coordinate, we integrate over it!)

\[
\Delta E_{\text{vert}} = \Delta E \cos d = \Delta E \cdot \frac{d}{\sqrt{r^2 + d^2}} = \frac{1}{4\pi \varepsilon_0} \cdot \frac{d r \delta}{(r^2 + d^2)^{3/2}} \Delta \phi
\]

Summing over all angular segments we get

\[
\sum \Delta \phi = 2\pi \quad \text{and}
\]

\[
\Delta E_{\text{ring}} = \frac{2\pi}{4\pi \varepsilon_0} \cdot \frac{d r \delta}{(r^2 + d^2)^{3/2}} \, dr
\]

To get the full field, we need to sum over rings, that is, integrate over \(r\):

\[
E_{\text{total}} = \int_0^\infty dr \cdot \frac{6}{2\varepsilon_0} \cdot \frac{r \cdot d\phi}{(r^2 + d^2)^{3/2}} = \frac{6}{2\varepsilon_0} \int_0^\infty d\phi \cdot \frac{1}{(1 + \frac{r^2}{d^2})^{3/2}} = \frac{6}{2\varepsilon_0}
\]

(radius changes from 0 to \(\infty\), as \(r \rightarrow \infty\)).

\[
E_{\text{plane}} = \frac{6}{2\varepsilon_0}
\]

does not depend on \(d\) at all!

In this way you will discuss a finite disc, similar to the way we discussed a line segment.